Confinement in Covariant Gauges

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ABSTRACT

We examine the weak coupling limit of Euclidean $SU(n)$ gauge theory in covariant gauges. Following an earlier suggestion, an equivariant BRST-construction is used to define the continuum theory on a finite torus. The equivariant gauge fixing introduces constant ghost fields as moduli of the model. We study the parameter- and moduli- space perturbatively. For $n_f \leq n$ quark flavors, the moduli flow to a non-trivial fixed point in certain critical covariant gauges and the one-loop effective potential indicates that the global $SU(n)$ color symmetry of the gauge fixed model is spontaneously broken to $U(1)^{n-1}$. Ward identities and renormalization group arguments imply that the longitudinal gauge boson propagator at long range is dominated by $n(n-1)$ Goldstone bosons in these critical covariant gauges. In the large n limit, we derive a nonlinear integral equation for the expectation value of large Wilson loops assuming that the exchange of Goldstone bosons dominates the interaction at long range in critical covariant gauges. We find numerically that the expectation value of large circular Wilson loops decreases exponentially with the enclosed area in the absence of dynamical fermions. The gauge invariance of this mechanism for confinement in critical covariant gauges is discussed.

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1 Introduction

Some exact results for the measure on the moduli space of supersymmetric theories have recently been obtained. Holomorphicity and the exactness of the one-loop β-function greatly restrict the low energy effective action of these models, and one can reach exact conclusions about their phase structure. When the number of matter fields is not too large, these non-abelian theories confine[1].

The absence of fundamental scalars is one of the obstacles which an investigation of nonperturbative effects in unbroken ordinary gauge theories has to confront. The moduli parameters in this case represent expectation values of composite operators and there is apparently no analog to the powerful constraint of holomorphicity for the effective potential. It was only recently observed that a translationally invariant quantization of ordinary gauge theories on compact manifolds does introduce moduli in a natural way. To eliminate normalizable zero modes of the ghosts that would lead to a vanishing partition function, the global $SU(n)$ symmetry of the gauge theory is treated equivariantly in a quantization which manifestly preserves the isometries of a compact space-time manifold. This equivariant BRST quantization[2] introduces a number of global moduli, one of which is a scalar in the adjoint representation with vanishing ghost number. A consistent covariant gauge fixing procedure thus introduces global moduli parameters even if the gauge symmetry is not broken and fundamental scalars are absent at the classical level.

We here investigate in detail the moduli-space of the covariantly quantized $SU(n)$ gauge theory on a torus and study the phase space of its unphysical degrees of freedom as a function of the gauge fixing parameters. We consider the most general renormalizable covariant gauges. They depend on three moduli and two gauge parameters: in addition to the usual covariant gauge parameter α related to the longitudinal gluon propagator, we also use the freedom to introduce a quartic ghost coupling in the gauge fixing functional[3]. Although we cannot compute the exact effective potential, the perturbative analysis reveals fixed points of the parameter and moduli space. These indicate that there are at least two asymptotically free phases of the unphysical degrees of freedom for $n_f \leq n$ quark flavors. Although no local physical order parameter distinguishes between the phases, we argue that a perturbative analysis of the model is consistent only *near* the phase transition. Since the moduli are introduced to avoid infrared problems in the gauge fixed action, the measure on this moduli space is gauge dependent. We however argue that it reflects non-perturbative properties of the gauge fixed model at fixed points in the parameter space.

The one-loop effective potential on the moduli space indicates that the phase transition is characterized by a condensation of ghost-antighost pairs in the adjoint representation. In the infinite volume limit this leads to a spontaneous breaking of the global $SU(n)$-symmetry to $U(1)^{n-1}$ of the equivariantly gauge fixed model. Although this pattern of spontaneous symmetry breaking is reminiscent of monopole condensation, the composite local scalar field in this case is gauge dependent and *not* a physical order parameter. One has to consider non-local Wilson[4]- or equivalently, magnetic[5]- loop operators as gauge

invariant order parameters for confinement.

The scale of symmetry breaking is found to be RG-invariant and proportional to Λ_{ASP} on a line in the parameter space. For a fixed number $n_f < n$ fermions in the fundamental representation, there are two fixed points on this line which merge for $n_f = n$. The corresponding covariant gauges we call critical covariant gauges (CCG), since perturbation theory appears to be selfconsistent in these gauges *also for unphysical degrees of freedom*.

Ward identities are used to show that the correlator of the composite local operator $\mathcal{O}^a(x) \propto f^{abc}\bar{c}^b(x)c^c(x)$ with the conserved color current in CCG has a Goldstone pole in the thermodynamic limit. We infer from this, that $\langle \partial\cdot A(x)\; \partial\cdot A(0)\rangle$ also has Goldstone behavior at large distances. This behavior is shown to be compatible with the Ward identities in CCG. In these gauges with quartic ghost interactions, Ward identities do not completely determine the longitudinal gluon propagator, but rather relate it to correlation functions in the ghost-antighost channel.

The contribution of the long-range longitudinal interaction to the expectation value of large Wilson loops is then obtained in leading order of n. Numerical solution of the non-linear integro-differential equation we derive in this approximation shows that the expectation value of large circular Wilson loops decreases exponentially with the enclosed area over 12 orders of magnitude.

The structure of this article is as follows. In section 2 we obtain the continuum action of the equivariantly quantized theory in general covariant gauges on a torus. This step is crucial, since it gauge fixes also the long wavelength excitations of the Yang-Mills theory without destroying the homogeneity and isotropy of the model. The equivariant construction introduces global fields, which we subsequently treat as moduli. In section 3 we study the renormalization flow behavior of the gauge parameters in such generalized covariant gauges. We locate the ultra-violet fixed point of the gauge parameter α, which is related to the longitudinal gluon propagator at large momentum transfers. In section 4 we determine the non-trivial fixed points of the moduli-space introduced by the equivariant gauge fixing. We find that the minimum of the effective potential is at a non-trivial value for one of these moduli and that the scale of this minimum is proportional to Λ_{ASP} in CCG. The critical exponent of the non-trivial moduli vanishes in these critical gauges. The one-loop effective potential also indicates that the global $SU(n)$-symmetry is broken to $U(1)^{n-1}$ in critical gauges. In section 5 we exploit the consequences of this symmetry breakdown using Ward identities of the model. These provide the connection between the perturbative indication that the $SU(n)$-symmetry of the model is spontaneously broken and long-range effects. We establish the existence of $n(n-1)$ unphysical Goldstone bosons associated with the global symmetry breaking by studying the $\mathcal{O}^a$-correlator with the conserved color current. Due to mixing between $\mathcal{O}^a$ and $\partial\cdot A$ by renormalization, the Goldstone bosons also dominate the long-range behavior of the longitudinal gluon-propagator in CCG. We show that this behavior is consistent with the Ward identities. In section 6 we use these results to approximately evaluate the expectation value of large circular Wilson loops in leading order of n. We derive a non-linear integral equation which resums the longitudinal interaction in leading order of n and we solve it numerically.

Section 7 is a summary and discussion of the results.

2 $SU(n)$ gauge theory in covariant gauges

To investigate phases of a quantum field theory, one has to control it in the thermodynamic limit of infinite space-time volume. We will study order parameters of an Euclidean $SU(n)$ gauge theory on a torus as the space-time volume of the torus becomes large.

A lattice regularization might appear to be the only reasonable definition of a gauge theory, since it does not depend on perturbation theory. Although this is a perfectly regularized statistical theory on a point manifold, the physical volume of any finite lattice shrinks exponentially as the coupling constant is tuned to the critical value. There is little hope of studying long-range effects numerically in the critical regime, and current simulations are restricted to a few fm^4 of physical volume. These numerical investigations suggest confinement, but deconfinement at a couple of GeV excitation energy cannot be excluded by these studies. Experimental evidence for the absence of colored (and/or fractionally charged) asymptotic states is considerably more restrictive and likely to remain so for some time.

A theoretical investigation of the thermodynamic limit of lattice gauge theory near its critical point is rather complicated. It has been pointed out[6], that the Gribov problem[7] cannot be ignored in a study of the lattice in this critical limit. A proposed effective description of lattice gauge theory at weak coupling introduces an additional parameter[6], which effectively constrains the configuration space to the fundamental modular region in the thermodynamic limit. It was shown[8] that a non-vanishing value for this parameter spontaneously breaks the BRST-symmetry of a lattice in Landau gauge[3].

We will not attempt to derive the thermodynamic limit of a lattice gauge theory here, but propose to investigate instead the consistency of perturbation theory in a covariantly gauge fixed continuum theory with presumably the same critical behavior as a lattice gauge theory. Let us first perturbatively define the continuum gauge theory on a compact space-time manifold such as a hypertorus. In order to effectively exploit the space-time symmetries of the thermodynamic limit, we consider only gauges which manifestly preserve *all* the isometries of the torus, i.e. general covariant gauges. The torus with periodic boundary conditions for the gauge and ghost fields is chosen for the following reasons:

- analytic continuation in the dimension D of the hypertorus T_D of perturbative expressions is straightforward. The perturbative dimensional regularization does not break the gauge- (or better BRST-) invariance explicitly. It allows one to efficiently extract the ultraviolet finite renormalized quantities in the limit $D \to 4$ and their critical exponents.

[3] The BRST-symmetry of a covariantly fixed *finite* lattice has to be broken, since the Euler character of the lattice gauge group vanishes and the gauge fixed partition function of any BRST-symmetric finite lattice model would therefore vanish as well[9].

- The space of gauge *orbits* of an $SU(n)$ theory on a hypertorus with periodic boundary conditions is connected. We thus only have to consider perturbation theory around a *single* classical gauge orbit. The absence of nontrivial topological sectors on the finite torus with periodic boundary conditions might oversimplify our study of the thermodynamic limit of this model. The non-perturbative effects we wish to investigate, however seem to be more easily described in a toroidal geometry of Euclidean space-time.
- An $SU(n)$ gauge theory defined on a torus resolves the topological ambiguities of the covariant gauge fixing procedure we employ[2]. These topological anomalies are present only on base manifolds with non-trivial 3-cycles.
- there is no topological obstruction to introducing fermions[10] in the fundamental representation on a torus with periodic boundary conditions. This would not be possible with twisted boundary conditions for the gauge field[5].
- The torus is a flat Euclidean space-time manifold. Translations and the associated Fourier - analysis are straightforward, and an explicit mode expansion is manageable.

These topological properties of a torus with periodic boundary conditions facilitate the perturbative analysis considerably. We will repeatedly take advantage of the translational and rotational invariance of gauge-dependent correlation functions in the thermodynamic limit of infinite space-time volume in covariant gauges.

2.1 Equivariant gauge fixing

Thermodynamic arguments for *gauge dependent* correlation functions are valid in *covariant* gauges, which by definition preserve the isometries of the base manifold. The Goldstone theorem in this case also gives information about the long-range behavior of *gauge-dependent* Green's functions.

It is well known that harmonic ghost modes pose an obstacle to the definition of a covariantly gauge-fixed partition function on compact manifolds[11]. These zero-modes of the Faddeev-Popov ghosts are a consequence of the invariance of covariant gauges with respect to *global* $SU(n)$ rotations. The zero modes are conventionally removed by fixing the global $SU(n)$ invariance at a particular space-time point[11]. Although this procedure preserves the BRST-algebra, such pointed gauges single out a space-time point, and thus break the translational invariance of gauge-dependent correlators explicitly. If an unbroken $SU(n)$ gauge theory indeed confines color, the usual argument that this point-defect can be disregarded in the thermodynamic limit of gauge dependent correlation functions does not necessarily hold, since color correlations must be strong and of long range. It is therefore not obvious that translational invariance of gauge dependent Green's functions, such as gluon- or ghost-propagators, is restored in the thermodynamic limit of pointed gauges. If translational and Euclidean rotational invariance of gauge dependent

correlators on the other hand cannot be explicitly maintained, the motivation for choosing a covariant gauge becomes questionable.

Recently a solution to the problem with harmonic ghost zero modes was proposed, which manifestly preserves the invariance of the gauge fixed theory with respect to *all* isometries of a compact manifold[2]. The global $SU(n)$-invariance is treated equivariantly in this construction and the usual field content of an $SU(n)$ gauge theory augmented by constant global ghosts whose canonical dimensions and ghost numbers are given in Table 1.

field	$A_\mu(x)$	$\psi_i(x)\&\bar{\psi}_i(x)$	$c(x)$	$\bar{c}(x)$	$b(x)$	ϕ	ω	σ	$\bar{\sigma}$	$\bar{\gamma}$	γ
dim	1	3/2	0	2	2	0	0	4	4	2	2
$\phi\Pi$	0	0	1	-1	0	2	1	-2	-1	0	1

Table 1. Dimensions and ghost numbers of the fields.

The action of the nilpotent BRST-operator s on the fields of Table 1 is[4]

$$\begin{aligned}
sA_\mu(x) &= D^A_\mu c(x) - [\omega, A_\mu(x)] \\
sc(x) &= -[\omega, c(x)] - \frac{g}{2}[c(x), c(x)] - \frac{1}{g}\phi \\
s\omega &= -\frac{1}{2}[\omega,\omega] + \phi \\
s\phi &= -[\omega,\phi] \\
s\bar{c}(x) &= -[\omega,\bar{c}(x)] + b(x) \\
sb(x) &= -[\omega, b(x)] + [\phi, \bar{c}(x)] \\
s\sigma &= -[\omega,\sigma] + \bar{\sigma} \\
s\bar{\sigma} &= -[\omega,\bar{\sigma}] + [\phi,\sigma] \\
s\bar{\gamma} &= -[\omega,\bar{\gamma}] + \gamma \\
s\gamma &= -[\omega,\gamma] + [\phi,\bar{\gamma}] \\
s\psi_i(x) &= -\omega\psi_i(x) - gc(x)\psi_i(x) \\
s\bar{\psi}_i(x) &= -\bar{\psi}_i(x)\omega - g\bar{\psi}_i(x)c(x)
\end{aligned} \tag{2.1}$$

where

$$D^A_\mu c(x) = \partial_\mu c(x) + g[A_\mu(x), c(x)] \tag{2.2}$$

is the usual covariant derivative for the adjoint representation.

It is straightforward to show that the BRST-operator defined above is nilpotent on any element of the graded algebra constructed from the fields of Table 1:

$$s^2 = 0 \tag{2.3}$$

[4]We will often suppress color indices. Except for $\psi_i, \bar{\psi}_i$, vectors of the two fundamental representations of $SU(n)$, all fields are traceless, anti-hermitian $n \times n$ matrices in this notation, and transform under the adjoint representation.

The constant ghost ω generates global gauge transformations of all the fields in the BRST algebra, except itself. One therefore can restrict observables to the equivariant cohomology Σ,

$$\Sigma = \{\mathcal{O} : \frac{\partial \mathcal{O}}{\partial \omega^a} = 0; s\mathcal{O} = 0, \mathcal{O} \neq s\mathcal{F}\} \tag{2.4}$$

where $\mathcal{F}$ is itself ω-independent. Since one is interested only in expectation values of gauge invariant functionals of $A, \bar{\psi}$ and ψ, the notion of *physical* observables can be further sharpened to functionals in the equivariant cohomology with vanishing ghost number[5]. The most general power counting renormalizable classical action S with the symmetry (2.1) depends on two gauge parameters α and $\beta = \alpha\delta$,

$$S = S_C[g, m_i; A, \psi_i, \bar{\psi}_i] + sW_{GF}[g, \alpha, \beta, \bar{\gamma}, \sigma; A, \bar{c}, c] \tag{2.5}$$

where S_C is the Yang-Mills action of an $SU(n)$ gauge theory with n_f quark flavors in the fundamental representation

$$S_C = \int_{\mathcal{T}} \frac{1}{2} \mathrm{Tr}\, F_{\mu\nu}(A) F_{\mu\nu}(A) + \sum_{i=1}^{n_f} \bar{\psi}_i(\not{D} + m_i)\psi_i \tag{2.6}$$

The Euclidean Dirac operator here is[6]

$$\not{D}\psi_i = \gamma_\mu(\partial_\mu + gA_\mu)\psi_i \tag{2.7}$$

and the field strength is related to the connection by

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{2.8}$$

Note that a CP-breaking term proportional to the Pontryagin number $\int_{\mathcal{T}} \mathrm{Tr}\, F \wedge F$ vanishes on a torus with periodic boundary conditions[7]. We therefore do not include such a term in (2.6). The discrete CP-symmetry in this sense is not broken on any finite torus.

The BRST-exact gauge fixing term of the action (2.5) was obtained in[2].

$$W_{GF} = 2\int_{\mathcal{T}} dx\, \mathrm{Tr}\, [\partial^\mu \bar{c}(x) A_\mu(x) - \frac{1}{2}\alpha \bar{c}(x) b(x) - \alpha\delta g \bar{c}(x)\bar{c}(x)c(x) + \frac{1}{g}\bar{\gamma}\bar{c}(x) + \frac{1}{g}\sigma c(x)] \tag{2.9}$$

For $\alpha > 0$ the Nakanishi-Lautrup field $b(x)$ in (2.5) can be eliminated using the equation of motion

$$b(x) = (\bar{\gamma}/g - \partial \cdot A(x))/\alpha - \delta g[\bar{c}(x), c(x)], \tag{2.10}$$

since observables do not depend on $b(x)$.

[5]To avoid global anomalies due to *large* gauge transformations, one could consider only *local* observables. However, the Chern-Simons term appears to be the only *gobal* physical observable with such a gauge- ambiguity[2]. The restriction to *local* observables is therefore not necessary on a torus.

[6]The Euclidean Dirac matrices γ_μ satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\mathbf{1}$.

[7]"Toron" sectors with non-vanishing Pontryagin number exist on a torus with twisted boundary conditions, which however can only be imposed in the absence of quarks in the fundamental representation[5].

A covariantly quantized $SU(n)$ gauge theory on a torus is thus described by the tree-level action

$$\begin{aligned} S_0 &= S_C + 2\int_{\mathcal{T}} dx \ \mathrm{Tr}\ \Big[\frac{1}{2\alpha}(\partial\cdot A(x))^2 + \delta\bar{c}(x)\, D^A\cdot\partial c(x) + (1-\delta)\bar{c}(x)\partial\cdot D^A c(x) + \\ &\quad +\alpha\delta(1-\delta)g^2\bar{c}(x)\bar{c}(x)c(x)c(x) - \delta[\bar{c}(x),c(x)]\bar{\gamma} + \frac{1}{2\alpha g^2}\bar{\gamma}^2 - \frac{1}{\alpha g}\bar{\gamma}\partial\cdot A - \\ &\quad -\sigma c(x)c(x) - \alpha(1-\delta)\phi\bar{c}(x)\bar{c}(x) - \frac{1}{g^2}\sigma\phi + \frac{1}{g}\gamma\bar{c}(x) + \frac{1}{g}\bar{\sigma}c(x)\Big]. \end{aligned} \tag{2.11}$$

The constant Grassmann ghosts γ and $\bar{\sigma}$ in (2.11) implement the constraints

$$\begin{aligned} \int_{\mathcal{T}} dx\ c(x) &= 0 \\ \int_{\mathcal{T}} dx\ \bar{c}(x) &= 0 \end{aligned} \tag{2.12}$$

which remove the (on a *finite* torus normalizable) constant modes of the ghost-fields in a translationally invariant fashion. In the framework of TQFT one can show that the partition function of the gauge-fixed quantum theory on a compact space-time manifold otherwise would vanish due to these zero-modes of the ghosts[11, 2, 9]. The surface term $\frac{1}{\alpha g}\bar{\gamma}\partial\cdot A$ in (2.11) vanishes on a finite torus with periodic boundary conditions. The absence of this term on a torus will eventually give rise to a non-trivial effective potential on the moduli-space. We retain this surface term in order to manifestly exhibit the BRST-symmetry of (2.11).

The action (2.11) is formally invariant under the action of $\tilde{s}$ defined by:

$$\begin{array}{rclrcl} \tilde{s}A_\mu(x) &=& D^A_\mu c(x) & \tilde{s}c(x) &=& -\frac{g}{2}[c(x),c(x)] - \frac{1}{g}\phi \\ \tilde{s}\bar{c}(x) &=& (\bar{\gamma}/g - \partial\cdot A(x))/\alpha - \delta g[\bar{c}(x),c(x)] & \tilde{s}\phi &=& 0 \\ \tilde{s}\sigma &=& \bar{\sigma} & \tilde{s}\bar{\sigma} &=& [\phi,\sigma] \\ \tilde{s}\bar{\gamma} &=& \gamma & \tilde{s}\gamma &=& [\phi,\bar{\gamma}] \\ \tilde{s}\psi_i(x) &=& -gc(x)\psi(x)_i & \tilde{s}\bar{\psi}_i(x) &=& -g\bar{\psi}_i(x)c(x) \end{array} \tag{2.13}$$

which is on-shell nilpotent only on the set of globally $SU(n)$ invariant functionals. Apart from an unorthodox transformation of the anti-ghost $\bar{c}$ and the presence of additional constant ghosts, $\tilde{s}$ is similar to the conventional BRST-transformation, and defines the equivariant cohomology of physical observables.

The constraints (2.12) are invariant with respect to (2.13) at points where

$$\int_{\mathcal{T}} dx\tilde{s}\bar{c}(x) = 0 \qquad \text{and} \qquad \int_{\mathcal{T}} dx\tilde{s}c(x) = 0 \tag{2.14}$$

Dividing by the finite volume of the torus and taking the thermodynamic limit, (2.14) relates the global bosonic ghosts $\bar{\gamma}, \phi$ to vacuum expectation values of composite dynamical fields

$$\langle\bar{\gamma}\rangle = \alpha\delta g^2\,\langle[\bar{c}(x),c(x)]\rangle \qquad \text{and} \qquad \langle\phi\rangle = -g^2\left\langle c^2(x)\right\rangle \tag{2.15}$$

The relations (2.15) show that the equivariant BRST-symmetry of the model is unbroken in the thermodynamic limit only at very specific values for the global ghosts. In what follows, we will see that the first of these relations in (2.15) is satisfied non-trivially in certain gauges and will explore the consequences.

The expectation value of an observable $\mathcal{O} \in \Sigma$ is formally given by the path integral

$$\langle \mathcal{O} \rangle = \mathcal{N} \int d\phi d\sigma d\bar{\sigma} d\bar{\gamma} d\gamma \int\!\!\int [\mathcal{D}\psi \mathcal{D}\bar{\psi}\, \mathcal{D}A\, \mathcal{D}c\, \mathcal{D}\bar{c}]\ \mathcal{O}\ \ \exp S_0, \tag{2.16}$$

which defines the perturbative loop expansion. We show in Appendix A that (2.16) is generally normalizable, i.e. that $\langle \mathbf{1} \rangle \neq 0$ for an $SU(2)$ group. The global bosonic ghosts ϕ, σ and $\bar{\gamma}$ introduced by the equivariant BRST-algebra are moduli parameters of the theory. We indicate in (2.16) that the integration over this finite dimensional moduli-space would usually be performed *after* the path integral over the dynamical fields. As far as the dynamical fields are concerned, the moduli are parameters of the action. Note that the moduli space possesses a trivial BRST-symmetry.

With finite fermion-masses m_i, and/or anti-periodic boundary conditions for the fields ψ_i and $\bar{\psi}_i$, only the constant modes of the connection $A_\mu(x)$ are potentially troublesome in a perturbative treatment of the theory. One can eliminate these constant modes by requiring twisted boundary conditions[5]. These boundary conditions however break the global $SU(n)$ invariance of the gauge theory explicitly[8] and effectively introduce external color flux[5]. Although this is arguably the most elegant treatment of the purely gluonic theory on a torus, twisted boundary conditions require the absence of fields in the fundamental representation. They also complicate the mode expansion and thus the infinite volume limit, especially since the space of gauge orbits in this case is disconnected. The global $SU(n)$ symmetry of the equivariantly gauge fixed theory is, as we shall see, *spontaneously* broken even for periodic boundary conditions at certain points in the parameter space. The theory is thus seen to react extremely sensitively to any external color flux.

The constant part a_μ of a field configuration $A_\mu(x)$ satisfying periodic boundary conditions

$$A_\mu(x) = \frac{1}{g} a_\mu + \tilde{A}_\mu(x) \qquad \text{with} \qquad \int_{\mathcal{T}} dx\ \tilde{A}_\mu(x) = 0 \tag{2.17}$$

can be treated as another moduli on which the perturbative expansion depends. Since we only evaluate correlation functions of local operators, the integration over this moduli space is governed by the corresponding effective action. Due to translational invariance of the constant modes, this effective action is proportional to the volume of space-time. In the infinite volume limit only the immediate vicinity of the absolute minimum (possibly degenerate) of the effective potential therefore contributes to the integration over the moduli-space. Contrary to supersymmetric gauge theories, we do not obtain the complete dependence of the effective potential on the moduli parameters. The assumption that perturbation theory is a good asymptotic expansion in conjunction with some physical criteria, however greatly constrains the minimum of the effective potential. Only the

[8] the equivariant construction is then in fact not necessary, since twisted boundary conditions also eliminate the constant ghost modes

minima which are stable fixed points can be consistently incorporated in a perturbative expansion of the theory, since moduli tend to flow to such fixed points as the coupling g becomes critical.

From a physical point of view, the minimum of the effective potential should occur for $a_\mu = 0$, since translational invariance implies that

$$\langle A_\mu(x) \rangle = \frac{1}{g} \langle a_\mu \rangle \, . \tag{2.18}$$

The discrete rotation group of a symmetric torus would be broken unless all $\langle a_\mu \rangle$, $\mu = 1, \ldots, D$ are equal. Equivariant observables such as $\mathrm{Tr}\, F_{\mu\nu} F_{\rho\sigma}$ or $\bar{\psi}_i(x)\gamma_\mu\psi(x)$ furthermore would register a breakdown of the $SO(4)$-symmetry in the continuum limit for $\langle A_\mu \rangle \neq 0$, unless this phase is gauge-equivalent to one with vanishing $\left\langle A'_\mu \right\rangle = 0$. A phase with nontrivial $\langle A_\mu \rangle$ is therefore acceptable only if

$$\langle a_\mu \rangle = U^\dagger(x) \partial_\mu U(x) \tag{2.19}$$

for some $U(x) \in SU(n)$ on the torus. In this case the theory is however physically equivalent to the one with $\langle a_\mu \rangle = 0$. Since the torus has only 1-cycles, (2.19) implies that $\langle a_\mu \rangle$ is abelian, i.e. $[\langle a_\mu \rangle , \langle a_\nu \rangle] = 0$. In Appendix A we find these abelian Gribov copies of the trivial vacuum $A_\mu = 0$. They are not removed by the equivariant construction on a torus. The contribution to $\langle A_\mu \rangle$ from these gauge copies however cancel. Gauge invariant local physical observables do not depend on which of these copies is selected, and we may as well perturbatively expand around the trivial configuration $\langle A_\mu \rangle = 0$.

The global $SU(n)$ symmetry is also spontaneously broken by a non-vanishing expectation value of any of the constant bosonic ghosts ϕ, σ or $\bar{\gamma}$ of the equivariant gauge fixing. None of the local functionals in the equivariant cohomology Σ are order parameters for a spontaneous breakdown of the global $SU(n)$ symmetry. We will see, the the *perturbative* evaluation of correlation functions in certain gauges is however sensitive to whether and how the global $SU(n)$-symmetry is broken. The situation is analogous to ordinary spontaneous symmetry breaking if one restricts the observables of the model to group invariants: the perturbative calculation of "invariant" correlations depends on whether the vacuum breaks the symmetry or not, but none of these invariant observables indicates which of the degenerate symmetry breaking vacua has been selected. Furthermore, the Goldstone bosons of the symmetry breaking *do not* lead to massless singularities in correlations of the invariant "observables". For such a restricted set of observables, the symmetry breaking is in a sense *hidden* and its effects appear to have a *dynamical* origin.

Non-vanishing $\langle \phi \rangle$ or $\langle \sigma \rangle$ implies that the ghost number is not conserved. The expectation value of the bosonic ghost $\bar{\gamma}$ however conserves the ghost charge and there is no a priori reason for it to vanish in the equivariant theory.

3 Fixed points of the gauge parameters

We begin by investigating the critical behavior of the dimensionless gauge parameters α and δ for $g \to 0$. For $\delta = 0$, the gauge parameter α was observed to flow to a non-trivial fixed point[12]. We will see that α flows to a fixed point α_∞ for a wide range of values of the gauge parameter δ.

Multiplicative renormalization constants relating bare (B) to renormalized (R) dynamical fields and parameters are defined as usual

$$\begin{array}{rclrclrcl} Z_2^{1/2}(\psi_R, \bar{\psi}_R) & = & (\psi_B, \bar{\psi}_B) & Z_3^{1/2} A_R & = & A_B & \tilde{Z}_3^{1/2}(c_R, \bar{c}_R) & = & (c_B, \bar{c}_B) \\ Z_g g_R & = & g_B & Z_\alpha \alpha_R & = & \alpha_B & Z_\delta \delta_R & = & \delta_B \end{array} \tag{3.1}$$

At the one loop level, δ does not renormalize[3]. Counter terms for the gluon polarization determine Z_3 as well as Z_α. Note that the usual Slavnov-Taylor identity for the longitudinal part of the gluon propagator[13] only requires equality of Z_α and Z_3 for $\delta = 0$. In generic covariant gauges the equation of motion of the Nakanishi-Lautrup field (2.10) also involves the composite operator $[\bar{c}(x), c(x)]$ and $Z_\alpha \neq Z_3$. To one loop this is confirmed by an explicit calculation of the renormalization constants of (3.1). In the MS-scheme one obtains in $D = 4 - 2\varepsilon$ space-time dimensions

$$\begin{aligned} Z_2 &= 1 - \tfrac{g^2\mu^{-2\varepsilon}(n^2-1)\alpha}{16\pi^2 n\varepsilon} + O(g^4) \\ \tilde{Z}_3 &= 1 + \tfrac{g^2\mu^{-2\varepsilon} n}{64\pi^2\varepsilon}(3-\alpha) + O(g^4) \\ Z_3 &= 1 + \tfrac{g^2\mu^{-2\varepsilon}}{16\pi^2\varepsilon}\left(\tfrac{13}{6}n - \tfrac{2}{3}n_f - \tfrac{\alpha n}{2}\right) + O(g^4) \\ Z_\alpha &= 1 + \tfrac{g^2\mu^{-2\varepsilon}}{16\pi^2\varepsilon}\left(\tfrac{13}{6}n - \tfrac{2}{3}n_f - \tfrac{\alpha n}{2} + \delta(1-\delta)\alpha n\right) + O(g^4) \\ Z_g &= 1 - \tfrac{g^2\mu^{-2\varepsilon}}{32\pi^2\varepsilon}\beta_0 + O(g^4) = 1 - \tfrac{g^2\mu^{-2\varepsilon}}{16\pi^2\varepsilon}\left(\tfrac{11}{6}n - \tfrac{1}{3}n_f\right) + O(g^4) \\ Z_\delta &= 1 + O(g^4) \end{aligned} \tag{3.2}$$

(The fact that $Z_3 = Z_\alpha$ also for $\delta = 1$ is not a consequence of the usual Slavnov-Taylor identity[14].) Note that the renormalization constant $\tilde{Z}_3$ of the dynamical ghosts does not depend on δ at the one-loop level. In (3.2) and most of the following we dropped the index (R) for renormalized quantities.

For $\alpha = \alpha_\infty$

$$\alpha_\infty(\delta) = \frac{13n - 4n_f}{3n(1 - 2\delta + 2\delta^2)} \tag{3.3}$$

Z_α is independent of the renormalization scale μ to order g^2. It is readily seen that α_∞ is the stable fixed point of the gauge parameter α which determines the asymptotic behavior of gauge dependent correlations.

The relation (3.1) between bare and renormalized gauge parameter implies for the renormalized α

$$\frac{\mathrm{d}\ \ln\alpha}{\mathrm{d}\ \ln\mu}+\frac{\mathrm{d}\ \ln Z_\alpha}{\mathrm{d}\ \ln\mu}=0 \tag{3.4}$$

in any gauge other than Landau gauges. Integration of (3.4) using the 1-loop expression for Z_α and

$$\frac{dg}{d\ln\mu}=\beta(g)=-\frac{g^3}{16\pi^2}\beta_0+\ldots\qquad\text{with}\qquad\beta_0=\frac{11}{3}n-\frac{2}{3}n_f \tag{3.5}$$

shows that the gauge parameter α approaches the fixed point α_∞ of (3.3) for sufficiently small coupling g as

$$\alpha-\alpha_\infty=C\alpha g^{\frac{13n-4n_f}{11n-2n_f}}\ (1+O(g^2)\ ) \tag{3.6}$$

where C is an integration constant related to the choice of gauge at a particular (small) value of the coupling. C is related to α in a similar fashion as Λ_{ASP} is related to the coupling g: it is the RG-invariant "gauge-parameter" which governs the asymptotic flow of α as $\mu\to\infty$. Note that the critical exponent in (3.6) is independent of δ and positive in the asymptotically free regime when

$$n_f<\frac{13}{4}n<\frac{11}{2}n \tag{3.7}$$

This implies that Landau gauge is *not* the asymptotically stable fixed point in the gauge parameter space if the number of families is not too large. Landau gauge becomes the stable fixed point only when the exponent in (3.6) is negative, i.e. for $\frac{11}{2}n>n_f>\frac{13}{4}n$ flavors. In the case of QCD this occurs only for $5-8$ families. Note also that α tends to α_∞ slower than $g^{13/11}$. The retained term of Z_α therefore dominates over the neglected terms of order g^4 in (3.4) at sufficiently small coupling. For $n_f=n$, α approaches α_∞ proportional to g. We remark that the fixed point α_∞ as well as the critical exponent in (3.6) tend to finite, non-vanishing values in the limit of large n, a case which will interest us. In general, (3.6) describes a surface in the space of continuous parameters $\alpha,\delta,g,n_f/n$ of the model. We include the ratio n_f/n as a parameter of the model, since asymptotic freedom for instance depends on its value.

For fixed n_f/n, the perturbative expansion of an $SU(n)$ gauge theory is selfconsistent in the vicinity of the curve in the coupling space g,α,δ corresponding to $g=0$, and $\alpha=\alpha_\infty(\delta)$ given by (3.3). The coupling constants g and α can be traded for the RG-invariants Λ_{ASP} and C. RG-invariant quantities only depend on Λ_{ASP}, C and δ, and *physical* gauge invariant correlations should furthermore not depend on the last two. An immediate consequence of this analysis is that the leading deep Euclidean behavior of the longitudinal gluon propagator is effectively described by the tree-level propagator in the gauge $\alpha_\infty(\delta)$: corrections to this asymptotic behavior vanish logarithmically for sufficiently large momentum transfers. At the fixed point α_∞ the term $\partial/\partial\alpha$ in the RG-equation only induces corrections that are *analytic* in g and which vanish for $g\to 0$.

It is perhaps surprising that a gauge parameter such as α depends on the renormalization scale at all. α was introduced as a coupling constant of the TQFT on the gauge

group[2] and the scaling-functions of a TQFT should vanish. The coupling constants of a TQFT are therefore not expected to depend on a renormalization scale. We do not have an exhaustive explanation for this apparent paradox. However, if one distinguishes between the definition of the TQFT and the *perturbative* calculation giving the apparent scale dependence of α, this may not be so paradoxical. We wish to point out that only *one* Gribov copy of a gauge field configuration contributes in the perturbative evaluation of a Green's function (the one in the vicinity of $A_\mu(x) = 0$). On the other hand, the conclusion that a parameter of the TQFT does not depend continuously on a scale requires the contributions from all stationary points, that is from *all* Gribov copies of a configuration. It is therefore conceivable that the asymptotic behavior of the longitudinal gluon propagator evaluated within a *single* Gribov region depends on the renormalization scale, whereas it would be scale independent, if the contribution from all other Gribov copies were considered as well (it is known that the "size" of a particular Gribov region is scale dependent[7]). Perturbatively one verifies only that gauge-dependent quantities evaluated in a *single* Gribov region, asymptotically require an adjustment of the gauge parameter with the scale μ. At the fixed point α_∞ this scale dependence vanishes and the contribution from other Gribov copies to the asymptotic behavior of gauge dependent Green's functions can be ignored. We find the existence of such a finite fixed point α_∞ remarkable.

The gauge invariant correlation functions one is ultimately interested in should not depend on whether all gauge copies of a configuration are taken into account or only one. Since the "number" of such copies does not change within a connected sector of the orbit space[2], the difference between the two evaluations of gauge independent Green's functions within a single topological sector can be absorbed in the normalization of the path-integral. The orbit space on a torus is connected, and one should therefore obtain the asymptotic expansion of physical correlation functions by perturbing around a *single* classical solution satisfying the gauge condition. (See[9] for a discussion of the dependence of the normalization on the sector for an $SU(2)$ gauge theory on S_4.)

We have found the fixed points in the parameter space. In the next section we consider fixed points of the moduli space. We will see that perturbation theory is consistent in a phase where the global $SU(n)$ symmetry is spontaneously broken to U^{n-1} for particular values of the gauge parameter δ. Although this consistency requirement for the *perturbative* expansion is gauge dependent, the fact that the global $SU(n)$ symmetry is broken to U^{n-1} perhaps is not. We will find that our moduli space is however consistent with such a symmetry breaking only on a subset of the fixed points in the parameter space. For other values of the parameters, non-perturbative configurations are probably relevant even asymptotically – just as contributions from other Gribov regions probably would stabilize α in gauges $\alpha \neq \alpha_\infty$. At particular points in the parameter space, perturbation theory indicates that the moduli space might be adequate to describe this symmetry breakdown and that other non-perturbative phenomena could be of sub-leading importance.

4 Fixed points of the moduli space

Perturbation theory is not entirely determined by the UV-fixed points of the parameters. It also depends on the phase, which usually is encoded in the expectation values of scalar fields. In the case of gauge theories these are composite operators, whose expectation values generally will not be described by the moduli of our model. We can however perturbatively examine whether this restricted moduli space allows for non-trivial expectation values at *particular* values of the model parameters.

Perturbation theory on the torus in covariant gauges depends on the moduli-space $\mathcal{M}$

$$\mathcal{M} = \{\bar{\gamma}, \phi, \sigma, a_\mu\} \tag{4.1}$$

The measure on this moduli-space is induced by integrating out the dynamical fields, which gives the effective potential for the moduli. In TQFT's the semi-classical measure on the moduli-space is *exact*, and in supersymmetric theories it is also severely constrained by holomorphicity[1].

In contrast to supersymmetric gauge theories, this measure on the moduli space will not be obtained exactly for the gauge theories that interest us. In a translationally invariant model, the effective action on the moduli space is however proportional to the space-time volume of the manifold. In the thermodynamic limit the integration over the moduli-space is thus effectively restricted to regions arbitrary close to absolute minima of the effective potential. To evaluate the integral over the moduli space in the thermodynamic limit in four space-time dimensions, we in principle only need to know the scale of these minima and the behavior of the effective potential in their immediate vicinity. If a compact group relates degenerate absolute minima of the potential and the observables are invariants under this group, it furthermore suffices to evaluate the moduli at *one* of the degenerate absolute minima only. The group is then spontaneously broken to the subgroup which leaves this minimum of the effective potential invariant.

Suppose for a moment that the effective potential of the asymptotically free theory in four dimensions has a unique RG-invariant non-trivial absolute minimum modulo group invariance. In the absence of dimensionful parameters, the scale κ associated with this minimum[9] is a function of the renormalization point μ and the dimensionless coupling constants g, α, δ only. Since the minimum is a fixed point of the moduli space and the theory is asymptotically free, κ must be proportional to Λ_{ASP} and its dependence on μ and g at weak coupling therefore is

$$\ln\frac{\kappa^2}{4\pi\mu^2} = -\frac{16\pi^2}{\beta_0 g^2} - \frac{\beta_1}{\beta_0^2}\ln\frac{\beta_0 g^2}{16\pi^2} + c_\kappa + O(g^2) \tag{4.2}$$

where

$$\begin{aligned} \beta_0 &= \tfrac{11}{3}n - \tfrac{2}{3}n_f \\ \beta_1 &= \tfrac{34}{3}n^2 - \tfrac{13}{3}nn_f + \tfrac{n_f}{n} \end{aligned} \tag{4.3}$$

[9]The canonical dimension of ϕ in (4.1) vanishes, but the effective potential conserves ghost number (although its minima need not to), and is thus a function of $\sigma\phi$ rather than ϕ itself.

are the first two coefficients of the β-function. Since the leading coefficient of $1/g^2$ in (4.2) cannot vanish, κ has to be the scale of the *one-loop* effective potential also. In fact, κ has to be associated with a non-trivial *absolute minimum* of the one-loop potential in the case we are considering, since the RG-invariant absolute minimum of the effective potential defines a mass scale. Conversely, a non-trivial absolute minimum of the one-loop effective potential which is RG-invariant and unique modulo group invariance *defines* an asymptotic mass scale which can be related to other definitions of Λ_{ASP} by a two-loop calculation[15]. It should be emphasized that an RG-invariant minimum of the effective potential on the moduli space gives an *asymptotic* scale, whose coefficients in the expansion (4.2) can be reliably calculated *perturbatively*.

The coefficient of the g^2 term in (4.2) is determined by the one-loop effective potential and *must* be $-(4\pi)^2/\beta_0$ at a fixed point of the moduli space. Depending on whether $\Lambda_{ASP} \gg \kappa$ or $\Lambda_{ASP} \ll \kappa$ at weak coupling, one otherwise concludes that the only fixed point of the moduli space is the trivial one with $\kappa/\Lambda_{ASP} = 0$ or the one at infinity. In the latter case the perturbative evaluation of the potential is certainly inconsistent, since it gives rise to an arbitrary large mass scale[10]. It should be emphasized that this first coefficient of the expansion (4.2) is entirely determined by the classical term of the effective potential and its *divergent* one-loop contribution. Since the effective potential is only UV-divergent, a perturbative calculation of this coefficient is reliable in asymptotically free gauge theories.

Whether κ is truly marginal for $g \to 0$ is determined by the coefficient of the $\ln g^2$ term in (4.2). It is given by the one-loop anomalous dimension of the order parameter. Since the effective potential is itself RG-invariant, the coefficients of the $1/g^2$- and $\ln g^2$-terms in fact are related by the RG-equation: near a fixed point in the parameter space, κ is of the form (4.2) only if the critical exponent of the moduli parameter vanishes. The relation between the first two coefficients in (4.2) also greatly facilitates the search for fixed points of the whole moduli space, since one merely needs to compute the critical exponents of the moduli on the curve (3.3) in the gauge parameter space. As shown in Appendix B, the anomalous dimension of a moduli is also determined by the divergence of the one-loop effective potential and the associated classical counterterm.

A two-loop calculation of the effective potential would in addition uniquely determine the coefficient c_κ in (4.2) and thus relate κ to any other asymptotic scale parameter quantitatively[15]. To qualitatively analyze the moduli space, the relation of κ to other asymptotic scale parameters is not required and will not be obtained here. We nevertheless emphasize, that a unique RG-invariant non-trivial absolute minimum of the effective potential on the moduli space is *perturbatively* related to other definitions of the asymptotic scale parameter in ordinary gauge theories.

It might appear that the converse is not necessarily true, and that a non-trivial absolute minimum of the one- and two-loop effective potential with the above properties does not necessarily imply the same for the full effective potential. The argument above how-

[10] the usual Λ_{ASP} should set the mass scale and the asymptotic expansion is not expected to depend on masses much larger than Λ_{ASP}.

ever shows that this would question the *validity* of perturbative results in ordinary gauge theories since non-perturbative effects would have to redefine the perturbative asymptotic scale[11]. Perturbation theory by itself ought to be adequate in determining critical exponents etc. of ordinary gauge theories, and the perturbative asymptotic scale can be given a physical interpretation in this case. This implies that the RG-invariant non-trivial absolute minimum of the two-loop effective potential which defines an *asymptotic* scale cannot simply be absent from the full effective potential. This argument for the existence of such a scale-defining absolute minimum also in the full effective potential does *not* mean that the two-loop effective potential is *exact* in ordinary gauge theories. It only implies that a RG-invariant scale set by the absolute minimum of the effective potential can be used to normalize the model also perturbatively and that the full effective potential must have a non-trivial absolute minimum, if the one-loop potential does and if this minimum is RG invariant. In the following we therefore only verify whether the one-loop effective potential on the moduli space has non-trivial RG-invariant absolute minima at fixed points of the parameter space.

Let us first assume that ghost number and angular momentum are conserved, i.e. that $\langle\phi\rangle = \langle\sigma\rangle = \langle a_\mu\rangle = 0$. In this case we need to consider only the dependence of the effective potential on the moduli $\bar\gamma$. Generalizing the calculation for an $SU(2)$ gauge group in[2] to $SU(n)$, we obtain in Appendix B the dependence of the one-loop effective potential on the eigenvalues $\lambda_i, i = 1, \ldots, n$ of the traceless $n \times n$ hermitian matrix $i\bar\gamma$

$$V_{1-loop} = \sum_{1\leq i<j\leq n} \frac{(\lambda_i - \lambda_j)^2\delta^2}{32\pi^2} \ln\left[\frac{(\lambda_i - \lambda_j)^2\delta^2}{e\kappa^4}\right] \tag{4.4}$$

Note that the full effective potential also depends only on eigenvalue differences, and that the logarithm in (4.4) is typical for the one-loop approximation. Crucial for the following is that (4.4) has a unique and non-trivial *absolute minimum* (modulo $SU(n)$) and the dependence of the associated scale κ in (4.4) on the renormalization point μ and the coupling constants. To this order in the loop expansion we obtain in Appendix B that,

$$\ln\frac{\kappa^2}{4\pi\mu^2} = -\frac{(4\pi)^2}{n\delta^2\alpha g^2} + 1 - \gamma_E + O(\ln g^2, g^2) \tag{4.5}$$

in the MS-scheme. Comparing with (4.2) we see that κ is an asymptotic scale only in gauges where

$$n\delta^2\alpha \underset{g\to 0}{\longrightarrow} \beta_0 \tag{4.6}$$

The intersections of this curve with $\alpha_\infty(\delta)$ of (3.3) occur for gauge parameters $\delta = \delta_\infty^\pm$

$$\delta_\infty^\pm = \frac{11n - 2n_f \pm \sqrt{2(n-n_f)(11n-2n_f)}}{9n} \tag{4.7}$$

[11] In $D = 2$ space-time dimensions the order parameter is dimensionless and the effective potential *does not* define an asymptotic scale and there is no spontaneous symmetry breaking[16]. In supersymmetric gauge theories in four dimensions, non-perturbative corrections are asymptotically important[17] and their contribution is required to unambiguously define an asymptotic scale. In theories which are not asymptotically free, such as ϕ^4, our argument obviously does not apply.

For a given value of n_f/n, the two points in the gauge parameter space $(\alpha_\infty(\delta_\infty^+), \delta_\infty^+)$ and $(\alpha_\infty(\delta_\infty^-), \delta_\infty^-)$, where the scale κ is comparable to Λ_{ASP} on the curve $\alpha = \alpha_\infty(\delta)$ define critical covariant gauges (CCG). Remarkably, we only find real intersections $\alpha_\infty(\delta_\infty^\pm), \delta_\infty^\pm$ in the asymptotically free regime when the number of quark flavors does not exceed the number of colors

$$n_f \leq n \tag{4.8}$$

Note that for $n_f > n$, $n\delta^2\alpha_\infty < \beta_0$ for any real value of δ. The scale κ in (4.5) is therefore irrelevant compared to Λ_{ASP} for $\frac{11}{2}n > n_f > n$. In the space of parameters $\delta, \alpha, n_f/n$ the condition (4.6) is valid on a surface which intersects the surface where (3.3) is satisfied on a critical curve. Projections of this curve onto the (δ,α)- and $(n_f/n,\delta)$-planes are shown in Fig. 1. Only on this critical curve in the parameter space, κ and Λ_{ASP} are *both* asymptotically relevant scales and proportional to each other. The perturbatively inaccessible region is shaded grey in Fig. 1 and given by $n\alpha_\infty\delta^2 > \beta_0$, since $\kappa \gg \Lambda_{ASP}$ in this case. We emphasize that (4.8) and (4.7) are consistency requirements for a *perturbative* evaluation. In other gauges (or when $n_f > n$) there is no non-trivial fixed point of the moduli $\bar{\gamma}$. In the grey region of Fig. 1 perturbation theory is not consistent and non-perturbative field configurations are perhaps important even asymptotically. In the white region of Fig. 1, the scale κ is irrelevant and the only fixed point of our moduli-space is the trivial one. The perturbatively interesting region is the phase boundary where $\kappa \sim \Lambda_{ASP}$.

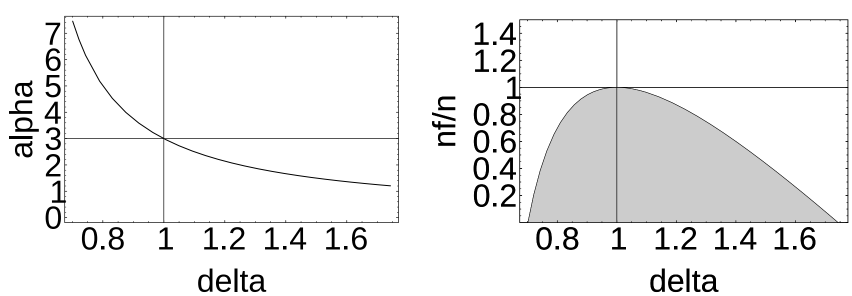

Fig. 1: Projection of the critical curve onto the (δ,α)- and $(\delta, n_f/n)$- planes. The two points on this critical curve for a fixed value of $n_f/n < 1$ define CCG. The perturbatively inaccessible region in the parameter space is shaded grey in the $(\delta, n_f/n)$-plane. The origin of the axes is at the critical point $(n_f = n, \delta = 1, \alpha = 3)$.

The gauge group of QCD is $SU(3)$ and there are $n_f = 6$ quark flavors. Since $n_f = 6 > 3 = n$ in this case, the perturbative analysis indicates that κ is an irrelevant asymptotic scale for QCD. However, only $2-3$ light quark flavors seem to be important for the low-energy dynamics of QCD and heavy quark flavors are furthermore unstable when flavor-changing weak interactions are included[12]. It is not inconsistent phenomenologically that κ is irrelevant at energies much above the J/Ψ threshold where more than 3 quark flavors are dynamical and where ordinary perturbation theory works reasonably well.

[12]The coupling to photons invalidates our perturbative analysis, which depends heavily on asymptotic freedom

To qualitatively describe the dynamics of QCD below a few GeV, an $SU(3)$ gauge theory with only 3 or even less dynamical quark flavors appears to be a reasonable model, especially when weak interactions are ignored. Heavy quark flavours appear to be adequately described by heavy quark expansions[18] around $\mu_f \sim \infty$. The purely gluonic theory with $n_f = 0$ and/or the large n limit of $SU(n)$ gauge theories are furthermore of theoretical interest, since Wilson's criterium[4] for absolute confinement is only valid in the absence of dynamical quarks. In these limiting cases, κ *is* an asymptotic scale in CCG.

Let us therefore explicitely verify that $\langle\bar\gamma\rangle$ is a truly marginal parameter in a (hypothetical) $SU(n)$ gauge theory with $n_f \leq n$ quark flavors in CCG. The critical exponent of $\bar\gamma$ vanishes if the anomalous dimension of $\bar\gamma^2$ is of order g^4. The renormalization constant $Z_{\bar\gamma^2}$ relating bare and renormalized moduli, $\bar\gamma_R^2 Z_{\bar\gamma^2} = \bar\gamma_B^2$, is calculated in Appendix B to order g^2. For arbitrary gauge parameters δ, α one obtains

$$Z_{\bar\gamma^2} = 1 - \frac{g^2\mu^{-2\varepsilon}n}{32\pi^2\varepsilon}\,(3 + \alpha(1-2\delta)) + O(g^4) \tag{4.9}$$

In CCG defined by the fixed points (3.3) and (4.7), the coefficient of g^2 in (4.9) vanishes and the critical exponent of $\bar\gamma^2$ thus vanishes as well. We thus have explicitly verified the relation between the two coefficients in (4.2) at fixed points of the parameter space. The non-trivial absolute minima of the effective potential defining the scale κ therefore do not depend on the value of the coupling g and define an asymptotic RG-invariant scale κ which is a physical mass scale in CCG[13]. As far as the perturbative evaluation of correlation functions invariant under the global $SU(n)$ symmetry in CCG is concerned, $\bar\gamma$ in (2.11) can be diagonalized and its eigenvalues treated as mass-parameters proportional to Λ_{ASP}^2 that do not depend on the coupling.

As remarked earlier, the absolute minimum of the one-loop effective potential (4.4) is unique modulo $SU(n)$ transformations. It is attained when none of the eigenvalue differences vanish, and the $SU(n)$ group is therefore maximally broken to U^{n-1}. Since the difference of two eigenvalues is proportional to the asymptotic scale parameter and furthermore RG-invariant, a variant of our previous argument indicates that the absolute minimum of the full effective potential also only possesses a U^{n-1} invariance in CCG. Higher order- and non-perturbative- corrections could only lift degeneracies among the eigenvalues and further break the symmetry. The difference of any two eigenvalues (for instance the lowest two) at the fixed point of the moduli space however defines a physical and asymptotic scale in CCG which cannot vanish in the full effective potential either.

We next verify our previous assumption that ghost number is conserved, i.e. that a non-trivial fixed point with $\langle\phi\sigma\rangle \neq 0$ would be *perturbatively* inconsistent. We only need to show that the critical exponent of $\sigma\phi$ does not vanish anywhere on the curve $\alpha_\infty(\delta)$ of (3.3) for n_f in the domain (3.7). A one-loop calculation similar to the previous one

[13] Note that for the extreme case $n_f = n$, the two fixed points coalesce at $\alpha_\infty = 3, \delta_\infty^\pm = 1$, a kind of Yennie-gauge[19] with nice infrared behavior of the quark propagator[20], and vanishing critical exponents of the ghost and gluon fields

gives for the renormalization constant $Z_{\sigma\phi}$ relating the bare and renormalized fields

$$Z_{\sigma\phi} = 1 + \frac{ng^2\mu^{-2\varepsilon}}{32\pi^2\varepsilon}\left(\alpha\delta(1-2\delta) - 3\right) + O(g^4) \tag{4.10}$$

The critical exponent of $\langle\sigma\phi\rangle$ therefore vanishes only if $\alpha_\infty\delta(1-2\delta) = 3$. Demanding also (3.3), this occurs only when the gauge parameter δ is a solution of the quadratic equation

$$\delta(1-2\delta)(13n - 4n_f) = 9n(1 - 2\delta + 2\delta^2) \tag{4.11}$$

(4.11) which has real solutions only for

$$n_f > \frac{-5 + 18\sqrt{2}}{4} n > \frac{13}{4} n \tag{4.12}$$

But $n_f > \frac{13}{4}n$ is outside the range (3.7) for which α approaches the fixed point $\alpha_\infty(\delta)$ asymptotically. Furthermore (4.10) is not of order g^4 at the fixed point $\alpha = 0$ for $n_f > \frac{13}{4}n$ either. The assumption that ghost number is broken therefore would be perturbatively inconsistent.

Consider finally the moduli a_μ, the constant parts of the gauge field. For a non-trivial fixed point with an anomalous dimension of order g^4 the term of order g^2 of the renormalization constant $Z_a = Z_3 Z_g^2$ would have to vanish. From (3.2) we see that this only occurs for $\alpha = -3$ – clearly an impossible value for the gauge parameter. When $\alpha > 0$, the anomalous dimension of a_μ is negative, and the only perturbatively stable fixed point is $a_\mu = 0$.

To summarize: the perturbative analysis of this section indicates that the global $SU(n)$-symmetry is spontaneously broken to $U(1)^{n-1}$ for $n_f \leq n$ quark flavors in CCG. Ghost number on the other hand appears to be conserved. The non-trivial expectation value $\langle i\delta\bar{\gamma}\rangle$ is characterized by a RG-invariant scale κ at two critical points of the parameter space

$$(g = 0, \alpha = \alpha_\infty(\delta_\infty^\pm), \delta = \delta_\infty^\pm) \tag{4.13}$$

given by (3.3), and (4.7). We next investigate qualitative implications for the low energy dynamics of the theory from the spontaneous breakdown of the global $SU(n)$ symmetry in CCG.

5 The broken phase

Since the equivariantly gauge fixed $SU(n)$ theory is *covariant* and translationally invariant in the infinite volume limit, spontaneously broken symmetries should give us access to the non-perturbative infrared dynamics of the model by Goldstone's theorem[21].

A non-trivial expectation value $\langle\bar{\gamma}\rangle \neq 0$ *does not* imply a breakdown of the equivariant BRST-symmetry[14]. $\langle\tilde{s}\bar{c}(x)\rangle$ in fact is the conjugate variable to $\bar{\gamma}$ in (2.11) in the sense

[14] The question of a global anomaly in the BRST-symmetry of the equivariantly quantized continuum model for certain base manifolds was addressed in[2].

that

$$\frac{\partial V_{eff}(\bar{\gamma})}{\partial \bar{\gamma}^a} = \langle \tilde{s}\bar{c}^a(x)/g\rangle_{\bar{\gamma}} = \left\langle \bar{\gamma}^a/(\alpha g^2) - \delta f^{abc}\bar{c}^b(x)c^c(x)\right\rangle_{\bar{\gamma}} \tag{5.1}$$

where we have used the equation of motion for $\bar{\gamma}$ and translational invariance. A *necessary* condition for an unbroken equivariant BRST-symmetry therefore is that $\bar{\gamma}$ is an extremum of V_{eff}. Conversely, if we require an unbroken BRST-symmetry (2.13), the possible expectation values for the moduli $\bar{\gamma}$ are solutions to the *gap equation*

$$\left\langle g^2\delta\alpha f^{abc}\bar{c}^b(x)c^c(x)\right\rangle_{\bar{\gamma}} = \bar{\gamma}^a \tag{5.2}$$

Note that (5.2) in general is a somewhat weaker statement than that $\langle\bar{\gamma}\rangle$ correspond to an *absolute minimum* of the effective potential. In the last section we saw that non-trivial RG-invariant solutions to (5.2) exist in CCG.

An superficially similar equation to (5.2) has previously been obtained in the thermodynamic limit of lattice gauge theory in Landau gauge as a necessary *horizon condition* for effectively constraining the gauge theory to the fundamental modular region[6]. Apart from the fact that (5.2) also introduces a mass scale via a consistency requirement, this gap equation differs from the *horizon condition* in many respects. Perhaps the most obvious difference is that the Landau-type gauges of[6] cannot be investigated in our way, since they do not correspond to perturbatively stable fixed points of the gluonic theory. Furthermore, a non-trivial solution to (5.2) breaks the global $SU(n)$ symmetry, which was assumed not to be broken in the lattice gauge theory[6]. Finally, it was shown in[8] that a non-trivial solution of the *horizon condition* spontaneously breaks the BRST-symmetry of the lattice gauge theory in the thermodynamic limit. The *gap equation* (5.2) in our case on the other hand is a minimal requirement for the equivariant BRST-symmetry of the model to remain unbroken.

Although (5.2) is only a necessary condition, we assume in the following that the equivariant BRST-symmetry is not broken in CCG. Perturbation theory gives no indication to the contrary and the breaking of BRST found in[2] for certain base manifolds does not occur for a torus. It should however be noted that the naive BRST-symmetry of covariant gauges[20] *is* broken – it would give a vanishing partition function and was the reason for restricting to an equivariant cohomology. It is therefore perhaps not so surprising that we find non-trivial fixed points of the moduli space with this equivariant construction in certain gauges.

(5.2) relates $\langle\bar{\gamma}\rangle$ to the expectation value of a *local* composite operator. To find the Goldstone bosons of the spontaneously broken symmetry, we first break the $SU(n)$ symmetry explicitly and extend the action (2.11) by a BRST-invariant local source term

$$S_0 \to S[\rho] = S_0 + 2\alpha\int \mathrm{Tr}\,\rho(x)\tilde{s}\bar{c}(x) \tag{5.3}$$

We now study local Ward-identities of this extended model in the limit of vanishing almost constant source $\rho(x)$ in the infinite volume limit.

To derive the relevant Ward-identity, consider the following local infinitesimal variation of the fields in the functional integral

$$
\begin{array}{rclrcl}
\delta A_\mu(x) & = & D^A_\mu\theta(x) = \partial_\mu\theta(x) + g[A_\mu(x),\, \theta(x)] & & & \\
\delta\psi_i(x) & = & -g\theta(x)\psi_i(x) & \delta\bar\psi_i(x) & = & g\theta(x)\bar\psi_i(x) \\
\delta c(x) & = & g[c(x),\, \theta(x)] & \delta\bar c(x) & = & g[\bar c(x),\, \theta(x)] \\
\delta\bar\gamma & = & \delta\gamma = \delta\bar\sigma = \delta\sigma = \delta\phi = 0 & & &
\end{array}
\tag{5.4}
$$

The transformation of the fermion- and gauge-fields in (5.4) is just an infinitesimal gauge transformation and thus only the gauge fixing part of the action (5.3) changes to first order in $\theta(x)$

$$
\begin{aligned}
\frac{\delta S[\rho]}{\delta\theta^a(x)} \; = \; & -\partial_\mu(J^a_\mu(x) + D^{ab}_\mu\rho^b(x)) + f^{abc}((\partial_\mu A^b_\mu(x)) + \alpha\delta g^2 f^{bde}\bar c^d(x)c^e(x))\rho^c(x) - \\
& -f^{abc}(\gamma^b\bar c^c(x) - g\delta\bar\gamma^b f^{cde}\bar c^d(x)c^e(x) + \bar\sigma^b c^c(x) - \\
& -\tfrac{g}{2}\sigma^b f^{cde}c^d(x)c^e(x) - \tfrac{g\alpha(1-\delta)}{2}\phi^b f^{cde}\bar c^d(x)\bar c^e(x)\,)
\end{aligned}
\tag{5.5}
$$

The variation (5.4) gives the color current

$$
\begin{aligned}
J^a_\mu(x) \; & = \; -\tilde s D^{ab}_\mu\bar c^b(x) - \alpha^{-1}f^{abc}\bar\gamma^b A^c_\mu(x) = \\
& = \; \alpha^{-1}D^{ab}_\mu(\partial\cdot A^b(x)) + g\delta f^{abc}c^b(x)D^{cd}_\mu\bar c^d(x) - g(1-\delta)f^{abc}\bar c^b(x)D^{cd}_\mu c^d(x)
\end{aligned}
\tag{5.6}
$$

without use of the equations of motion. The current (5.6) does not explicitly depend on the moduli and in fact is just the usual one for gauges with quartic ghost interaction[3]. Contrary to the standard case however, this current is *not* $\tilde s$-exact for $\bar\gamma \neq 0$. Note that the constant ghosts do not transform under the *local* variation (5.4) and that the change of the action is therefore *not* proportional to the divergence of the color current (5.6). This somewhat complicates the Ward identities, but we will see that the additional terms are irrelevant at vanishing momentum transfer. Because the gauge symmetry is not anomalous, the measure of the path integral is invariant under the change of variables (5.4). We thus have the Ward-identity

$$
\left\langle \frac{\delta\mathcal{O}}{\delta\theta^a(x)} \right\rangle_\rho + \left\langle \mathcal{O}\frac{\delta S[\rho]}{\delta\theta^a(x)} \right\rangle_\rho = 0
\tag{5.7}
$$

where $\langle\ldots\rangle_\rho$ indicates that the expectation value is for fixed source $\rho(x)$ and $\mathcal{O}$ is some functional of the fields. With $\mathcal{O} = \mathcal{O}^a(x)$

$$
\mathcal{O}^a(x) = g\alpha\delta f^{abc}\bar c^b(x)c^c(x)
\tag{5.8}
$$

(5.7) at vanishing momentum transfer in the limit of infinite space-time volume becomes

$$
\begin{aligned}
& \int d^4y\ \left\langle \mathcal{O}^a(x)\partial\cdot(J^b(y) + D^{bc}\rho^c(y))\ \right\rangle_\rho = \left\langle g f^{acb}\mathcal{O}^c(x)\right\rangle_\rho + \\
& + g\int d^4y f^{bcd}\rho^c(y)\frac{\delta}{\delta\rho^d(y)}\left\langle\mathcal{O}^a(x)\right\rangle_\rho \\
& + \left\langle \mathcal{O}^a(x)\ g f^{bcd}\left(\bar\gamma^c\frac{\partial}{\partial\bar\gamma^d} + \gamma^c\frac{\partial}{\partial\gamma^d} + \bar\sigma^c\frac{\partial}{\partial\bar\sigma^d} + \sigma^c\frac{\partial}{\partial\sigma^d} + \phi^c\frac{\partial}{\partial\phi^d}\right) S[\rho] \right\rangle_\rho
\end{aligned}
\tag{5.9}
$$

Since $\mathcal{O}^a(x)$ does not explicitly depend on constant ghosts, the last term in (5.9) is proportional to equations of motion of the moduli and vanishes. In the limit $\rho(x) \to 0$ of a slowly varying source, (5.9) requires that

$$\int d^4x \,\left\langle \partial\cdot J^a(x)\; \mathcal{O}^b(y)\right\rangle = \int d^4x\,\left\langle \partial\cdot J^a(x)\; g\alpha\delta f^{bcd}\bar{c}^c(y)c^d(y)\right\rangle = f^{abc}\bar{\gamma}^c \tag{5.10}$$

where we used (5.1) to express the expectation value of the operator $\langle\mathcal{O}^a(x)\rangle_{\rho\to 0}$ in terms of $\bar{\gamma}^a$. Since the RHS of (5.10) does not vanish at the non-trivial minimum of the effective potential in CCG, (5.10) implies a long range behavior of the correlation function

$$\left\langle J^a_\mu(x)\; \mathcal{O}^b(0)\right\rangle \underset{x^2\to\infty}{\longrightarrow} f^{abc}\bar{\gamma}^c \frac{x_\mu}{2\pi^2 x^4} \tag{5.11}$$

which is the signature of massless Goldstone modes. Note that the last step assumes translational *and* Euclidean $SO(4)$ invariance of the thermodynamic limit.

The relation (5.11) between $\langle\bar{\gamma}\rangle$ and the correlation function of the composite operator $\mathcal{O}^a(x)$ of (5.8) with the color current (5.6) does not appear to be RG-invariant, since $\mathcal{O}^a(x)$ generally mixes with $\partial\cdot A$. It is however straightforward to check that the divergences on the LHS of (5.11) cancel to order g^2. We have observed earlier that $Z_{\bar{\gamma}^2} = 1 + O(g^4)$ *precisely* at the fixed points (4.13) defining CCG. The relation (5.11) is therefore consistent in the weak coupling regime of CCG.

However, divergences in the correlation function (5.11) only cancel to first order in the loop expansion for the *combination* of composite operators in (5.6). The mixing between individual terms in J^a implies that the Goldstone pole must also be present in every correlation function of $\mathcal{O}^a(x)$ with any composite operator appearing in the color current J^a and in particular in the 1PR part of J^a

$$\left\langle \partial_\mu\partial\cdot A^a(x)\; \mathcal{O}^b(0)\right\rangle \underset{x^2\to\infty}{\longrightarrow} -\frac{2x_\mu Z(\mu)}{x^4} f^{abc}\bar{\gamma}^c \tag{5.12}$$

The residue $Z(\mu)$ in (5.12) depends on the renormalization point μ even at arbitrary weak coupling – only the residue in (5.11) of the Goldstone contribution to the correlation of $\mathcal{O}^a$ with the full current J^a_μ is RG-invariant in CCG in the limit $\mu\to\infty$. $Z(\mu)$ however cannot vanish for arbitrary values of the renormalization scale, because the operator $\partial_\mu\partial\cdot A^a$ is necessary for a cancelation of the one-loop divergences on the LHS of (5.11). A coupling to the Goldstone modes is therefore inevitably generated in (5.12) by a change of the renormalization scale.

In the translationally invariant theory, (5.12) requires a long range behavior of the "mixed" propagator of the form

$$\left\langle \partial\cdot A^a(x)\; \mathcal{O}^b(0)\right\rangle \underset{x^2\to\infty}{\longrightarrow} \frac{Z(\mu)}{x^2} f^{abc}\bar{\gamma}^c \tag{5.13}$$

We now argue that a long-range behavior of the longitudinal gluon propagator

$$\left\langle \partial\cdot A^a(x)\; \partial\cdot A^b(0)\right\rangle \underset{x^2\to\infty}{\longrightarrow} \frac{Z_L(\mu)}{x^2} f^{acd} f^{bce}\bar{\gamma}^d\bar{\gamma}^e \tag{5.14}$$

and of the $\mathcal{O}^a$-propagator

$$\left\langle \mathcal{O}^a(x)\ \mathcal{O}^b(0) \right\rangle \underset{x^2\to\infty}{\longrightarrow} -\frac{Z_L(\mu)}{x^2} f^{acd} f^{bce} \bar{\gamma}^d \bar{\gamma}^e \tag{5.15}$$

also is a natural consequence of the massless Goldstone excitations. In the case of covariantly quantized gauge theories this coupling to Goldstone excitations is unfortunately not as inevitable as in a theory with a positive definite Hilbert space.

The residues in (5.14) and (5.15) are related by the $\tilde{s}$-symmetry (2.13), which implies that

$$\left\langle \tilde{s}\bar{c}^a(x) \tilde{s}\bar{c}^b(y) \right\rangle = -\left\langle \bar{c}^a(x)\ \tilde{s}\tilde{s}\bar{c}^b(y) \right\rangle = \alpha^{-1} \delta^{ab} \delta^4(x-y) \tag{5.16}$$

In gauges $\delta = 0$ this is the Slavnov-Taylor identity which determines the longitudinal part of the gluon propagator completely. In general gauges with $\delta \neq 0$, (5.16) only relates connected Green's functions and in particular requires the absence of Goldstone excitations in the combination

$$\begin{aligned} 0 = \alpha^2 \left\langle \tilde{s}\bar{c}^a(x) \tilde{s}\bar{c}^b(0) \right\rangle_{x^2>0} \ = \ & \Big[\left\langle (\mathcal{O}^a(x) - \langle \mathcal{O}^a \rangle)\ (\mathcal{O}^b(0) - \langle \mathcal{O}^b \rangle) \right\rangle + \left\langle \partial\cdot A^a(x)\ \partial\cdot A^b(0) \right\rangle \\ & + \left\langle (\mathcal{O}^a(x) - \langle \mathcal{O}^a \rangle)\ \partial\cdot A^b(0) \right\rangle + \left\langle \partial\cdot A^a(x)\ (\mathcal{O}^b(0) - \left\langle \mathcal{O}^b \right\rangle) \right\rangle \Big]_{x^2>0} \end{aligned} \tag{5.17}$$

for $x^2 > 0$.

The color structure of (5.13) leads to a cancelation of the long range parts of the last two terms in (5.17). The long range correlations of the first two terms also have to cancel for (5.17) to hold. Note that for $\delta \neq 0$, (5.17) does not demand vanishing residue Z_L, because the metric of the Hilbert space of the covariant gauge theory is not positive definite.

States of negative norm in covariant gauges however also prevent us from proving in the usual way that Z_L in (5.14) does *not* vanish. A non-trivial cancelation of the Goldstone poles between the first two terms of (5.17) however only requires that the vertex $\Gamma^{[ab]}_{\mathcal{O}\,\partial A}$ between the scalar $c\bar{c}$-mode and longitudinal gluons with an antisymmetric color structure does not vanish for $p^2 \to 0$. We perturbatively verify that this is the case in the broken phase of the $SU(2)$-model in Appendix C. The one-loop calculation of this vertex gives

$$\Gamma^{[ab]}_{\mathcal{O}\,\partial A}(p^2 \to 0) = \varepsilon^{abc} \bar{\gamma}^c \frac{3g^2\delta^2}{32\pi\kappa^2} \tag{5.18}$$

Note that this dimensionless antisymmetric coupling arises only when the global $SU(2)$ symmetry is spontaneously broken. A vanishing of the full $\mathcal{O}\,\partial A$-vertex at zero momentum and weak coupling would require a cancelation of this leading contribution from non-perturbative corrections that are of order $1/g^2 \propto \ln(\Lambda^2/\mu^2)$. Such a dependence on the renormalization scale μ is generally induced only by divergences. Since the vertex (5.18) is not present in the action (2.11) and the model is renormalizable, the $\mathcal{O}\,\partial\cdot A$-vertex antisymmetric in color is finite to all orders. We thus do not expect a cancelation of the lowest order vertex (5.18) at weak coupling.

The color structure of (5.18) guarantees the cancelation of Goldstone poles in the Ward identity (5.17). Since the adjoint representation is the only one common to the antisymmetric tensor product of two adjoint representations and the symmetric tensor product of any number of adjoint representations, the corresponding vertex for $SU(n)$ can only be proportional to $f^{abc}P^c(\bar{\gamma})$, where $P^c(\bar{\gamma}) = p_1\bar{\gamma}^c + p_2 d^{def}\bar{\gamma}^d\bar{\gamma}^e d^{cfg}\bar{\gamma}^g + \ldots$ is an odd and totally symmetric polynomial in $\bar{\gamma}$ in the adjoint representation of the group. That the color structure of the Goldstone contribution to $\langle \partial\cdot A(x)\, \partial\cdot A(0)\rangle$ is of the form (5.14) then follows from (5.13) and Bose-symmetry. The real, symmetric and positive semidefinite $(n^2-1)\times(n^2-1)$ matrix $M^{ab} = f^{adc}\bar{\gamma}^c f^{bde}\bar{\gamma}^e$ has exactly $(n-1)$ vanishing eigenvalues when $\bar{\gamma}$ breaks the $SU(n)$-group to $U(1)^{n-1}$. Thus the $n(n-1)$ positive eigenvalues of M^{ab} and the number of expected Goldstone modes correspond. Of some importance for the next section is that the vertex $\Gamma_{\mathcal{O}\,\partial A}$ is of order n^0 for large n, since the expectation value $\langle\bar{\gamma}^a\rangle$ is of order n^0 and insertions of $\langle\bar{\gamma}\rangle$ therefore do not change the leading order in n of a diagram. The one-loop diagram shown in Fig. 5, which we evaluated for $SU(2)$ in Appendix C, is planar and thus of order n^0 in $SU(n)$.

Finally note that the critical exponent of the residue $Z_L(\mu)$ in (5.14) is given by the anomalous dimension of the gluon field of order g^2, since the anomalous dimension of $\bar{\gamma}$ is of order g^4 in CCG.

Although indicative, we are well aware that these arguments for a Goldstone contribution to the longitudinal gluon propagator in CCG are not conclusive. In the next section we therefore present further evidence for a long range behavior of the *longitudinal* gluon propagator of the form (5.14): assuming that the dominant long-range contribution to a Wilson loop in CCG comes from this Goldstone contribution to the longitudinal gluon interaction, the expectation value of large Wilson loops is found to decay exponentially with the enclosed area in the planar gluonic theory.

6 The Wilson loop in CCG

The order parameter of confinement proposed by Wilson[4] is the expectation value of the non-local functional $W(\mathcal{C})$

$$W(\mathcal{C}) := \mathrm{Tr}\,\mathcal{P}\,\exp\left[g\oint_{\mathcal{C}} dx_\mu A_\mu(x)\right] \tag{6.1}$$

where $\mathcal{P}$ denotes path ordering and $\mathcal{C}$ is a generic smooth path. In a confining gluonic theory ($n_f = 0$) the expectation value of (6.1) should decay exponentially with the area of the minimal surface enclosed by $\mathcal{C} = \partial D$

$$\ln\langle W(\partial D)\rangle \underset{\mathrm{Area}(D)\to\infty}{\longrightarrow} -\sigma\mathrm{Area}(D) \tag{6.2}$$

where the constant of proportionality σ is the string tension. This should in particular be the case for discs D with large Euclidean radius R.

$\langle W(\partial D)\rangle$ is a gauge- as well as RG-invariant observable for any $SU(n)$ gauge theory. It suffices to show that the string tension σ does not vanish for large n, i.e. for the subset of planar diagrams[22] in the expansion of the Wilson loop. A cancelation between planar and non-planar contributions to the expectation value precisely for $n=3$, the gauge group of interest, would be unfortunate. We will therefore only study planar contributions to the expectation value (6.2) of circular Wilson loops such as the one shown in Fig. 2.

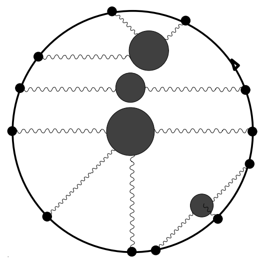

Fig. 2: A planar gluonic contribution to a circular Wilson loop.

Although the leading order in n of $\langle W(\partial D)\rangle$ is gauge invariant, the contributions from planar connected gluonic Green's functions to a diagram such as Fig. 2 *are* gauge dependent *individually*. It is therefore possible, that some of these contributions to the expectation value are negligible compared to others in *certain* gauges when the area of the loop is large. In the preceding sections we argued that the longitudinal gluon propagator in CCG is of particularly long range due to Goldstone modes[15]. The planar connected Green's functions contributing to large Wilson loops naturally fall into two categories in CCG: i) those which are dominated by the Goldstone poles at long range and ii) those which aren't. Contributions of type i) at large distances can be regarded as longitudinal gluon exchanges with an effective coupling $G(\mu)$ to the loop. $G(\mu)$ coincides with the gauge coupling $g(\mu)$ in lowest order of the weak coupling expansion only and is also of order $1/\sqrt{n}$. (The order $g^3(\mu)$ corrections to $G(\mu)$ are obtained in Appendix D. One finds that $G(\mu)$, the effective coupling of a longitudinal gluon to the Wilson loop, and the perturbative coupling $g(\mu)$ differ in their dependence on the renormalization scale μ.) In critical gauges contributions of type i) give rise to a long range interaction

$$v^{ab}_{\mu\nu}(x^2\sim\infty) = G^2(\mu)\left\langle A^a_\mu(x)\ A^b_\nu(0)\right\rangle\Big|_{Goldstone} \tag{6.3}$$

from the exchange of $n(n-1)$ Goldstone bosons. Because the $SU(n)$ group is broken to $U(1)^{n-1}$, the number of Goldstone bosons is of order n^2. To show that the interaction (6.3) is also of leading order in n, however requires that their average residue does not vanish in the limit of large n. As remarked earlier, it is evident from the action (2.11) that insertions of $\bar\gamma$ are of order n^0 and thus do not change the leading order in n of a diagram. Thus the effective potential (4.4) is of order n, but its minimum, $\langle\bar\gamma\rangle$ as well as the scale κ are of order n^0. Since all eigenvalues of $\langle\bar\gamma\rangle$ differ and each is of order n^0, the sum of their squares therefore grows like n, that is

$$\bar\gamma^a\bar\gamma^a = -2\mathrm{Tr}\,\bar\gamma^2 = 2\sum_{i=1}^{n}\lambda_i^2 = O(n) \tag{6.4}$$

[15]With $n_f=0$ the two CCG are at gauge parameters $\delta^\pm=\frac{11\pm\sqrt{22}}{9}$ and $\alpha^\pm=\frac{13(13\pm2\sqrt{11})}{81}$

because at most one of the eigenvalues vanishes. To leading order in n we can therefore replace the color dependence of (5.14) by the average

$$\left\langle \partial\cdot A^a(x)\ \partial\cdot A^b(0)\right\rangle \underset{x^2\to\infty}{\longrightarrow} \frac{Z_L(\mu)}{x^2}\delta^{ab}\frac{\bar{\gamma}^c\bar{\gamma}^c}{n} + O(1/n) \tag{6.5}$$

Due to (6.4), the Goldstone contribution to the longitudinal propagator is of $O(1)$, if the residue Z_L of the Goldstone modes is of order n^0. The previous argument that the coupling Z_L does not vanish is also valid in leading order of n (since the Ward identity (5.10) is of leading order and the leading contributions to the vertex $\Gamma^{[ab]}_{\mathcal{O}\,\partial A}$ shown in Fig. 5 are planar). As far as color counting is concerned, the exchange of Goldstone modes in CCG therefore contributes to the Wilson loop in leading order of the $1/n$ expansion.

Integrating (6.5) we obtain the long-range behavior of the longitudinal gluon propagator to $O(1)$

$$\left\langle A^a_\mu(x)\ A^b_\nu(0)\right\rangle \underset{x^2\to\infty}{\longrightarrow} -Z_L(\mu)\delta^{ab}\frac{\bar{\gamma}^c\bar{\gamma}^c}{32n}\partial_\mu\partial_\nu x^2\ln(b(\mu)x^2) \tag{6.6}$$

where we have assumed the least singular behavior for $x^2\to\infty$ compatible with (6.5). The integration constant $b(\mu)$ cannot be determined at this stage. We will numerically find that $b(\mu)$ is related to $Z_L G^2$ if the physical spectrum is unitary. At present we only remark that changing the parameter b alters the propagator by a constant and therefore changes its long range behavior. The Goldstone pole in the correlator (6.5), which was inferred from the Ward identities, only determines the *leading* long-range behavior of the propagator which does not depend on the associated scale b. Note however that an explicit solution of the model, would give the long range part of the gluon propagator up to a physical scale and determine the associated dimensionless parameter Z_L/b uniquely in terms of the dimensionless coupling constants.

Inserting (6.6) in (6.3) we arrive at a parameterization of the long range 2-point interaction in leading order of n

$$v^{ab}_{\mu\nu}(x^2\sim\infty) = \frac{\delta^{ab}}{n}K^2\partial_\mu\partial_\nu x^2\ln(sK^2x^2/e^2) \tag{6.7}$$

in terms of K and s which are related to G^2Z_L respectively $b/(G^2Z_L)$. In (6.7) s is a dimensionless parameter and K is an inverse length. In Appendix D we verify that these parameters approach renormalization group invariants for $\mu\to\infty$. To establish this, we consider the *perturbative* planar 1-loop corrections of Fig. 6 to the effective long-range exchange (6.7) within a Wilson loop. The explicitly calculation of Appendix D shows that the UV-divergences associated with an evaluation of diagrams 6a+b precisely cancel the renormalization factor for the propagator and coupling to order g^2. No UV-subtractions of order g^2 are therefore necessary to render the contribution to the expectation value of the Wilson loop from effective planar longitudinal 2-point exchanges finite. The interaction (6.7) therefore does not depend on the renormalization scale μ as $\mu\to\infty$.

In Appendix E we prove that planar contributions to the Wilson loop generated by an effective 2-point longitudinal interaction v are resummed by a nonlinear integral equation.

In the case of a circular Wilson loop of radius R this integral equation is of the relatively simple form

$$W_R(\theta) = \mathbf{1} + \int_0^\theta d\theta' \int_0^{\theta'} d\theta'' v_R(\theta'') W_R(t'') W_R(\theta' - \theta'') \tag{6.8}$$

In (6.8) the color traces have been performed and the expectation value of the circular Wilson loop $W(\partial D_R)$ of large radius R with the above approximations is

$$\langle W(\partial D_R)\rangle \overset{R\sim\infty}{\sim} n W_R(2\pi) \ . \tag{6.9}$$

The effective interaction $v_R(\theta)$ in (6.8) is related to (6.7) by

$$\begin{aligned} v_R(\theta) &= \frac{\partial x_\mu}{\partial \theta_x}\frac{\partial y_\nu}{\partial \theta_y}\frac{1}{n}\mathrm{Tr}\, t^a t^b v^{ab}_{\mu\nu}(x(\theta_x) - y(\theta_y))\Big|_{\theta=\theta_x-\theta_y} + O(1/n^2) \\ &= \frac{1}{2}\frac{\partial^2}{\partial\theta^2}(KR)^2 \sin^2(\theta/2)\ln(s(KR/e)^2\sin^2(\theta/2)) \\ &= (KR)^2\left[1 + \cos(\theta)\ln(s(KR)^2\sin^2(\theta/2))\right] \end{aligned} \tag{6.10}$$

since the distance $(x-y)^2 = 4R^2\sin^2((\theta_x-\theta_y)/2)$ of two points on the loop depends only on the radius R and the angular difference $\theta = \theta_x - \theta_y$ between them. The interaction (6.10) depends parametrically on the dimensionless area $(KR)^2$ of the loop and the dimensionless parameter s.

We numerically solved the nonlinear integro-differential equation

$$\frac{\partial}{\partial\theta}W_R(\theta) = \int_0^\theta d\theta'\, v_R(\theta') W_R(\theta') W_R(\theta - \theta')\,, \quad W_R(0) = 1 \tag{6.11}$$

which is equivalent to (6.8). For small values of $\theta \ll (KR)^2$ (6.11) was solved iteratively and we then used these starting values to extend the solution to $\theta = 2\pi$ by a predictor-corrector algorithm based on the Adams-Bashforth-Moulton scheme. The numerical integrations where performed with a modified Rhomberg algorithm[23]. We found the predictor-corrector method appropriate in this case, since the integral in (6.11) depends on all previous values for the function W_R. $W_R(\theta)$ itself is a reasonably smooth function which oscillates a few times in the interval $[0, 2\pi]$. Special care was taken to correctly take into account the logarithmic (but integrable) singularities of v_R at $\theta = 0$ and $\theta = 2\pi$. The algorithm is exceptionally stable and we quote results which we believe[16] to be accurate to $\sim 10^{-12}$. The results of these numerical simulations are summarized in Fig. 3. For a particular value $s = s_{crit.} = 1.195$, we numerically find that the expectation value of the Wilson loop decreases exponentially with the dimensionless area $(KR)^2$ over 12 orders of magnitude. The results for the Wilson loop with neighboring values $s = 1.17$ and $s = 1.25$ are also shown in Fig. 3 for comparison. We observe that for $s = 1.17 < s_{crit.}$ the logarithm of the Wilson loop decreases faster than linear with the area. Such a behavior of the expectation value is excluded by causality[24]. For $s = 1.25 > s_{crit.}$ we also observe a violation of causality – for large values of the area, the expectation value

[16] We used up to 10^4 integration points in the interval $[0, 2\pi]$ and the predictor-corrector algorithm is of fourth order.

increases again. This behavior is more marked at even larger values of s, as can be seen for $s = 2.0$ in Fig. 3. At $s = s_{crit.}$ the oscillations which eventually cause the expectation value of the Wilson loop to become negative are of order 10^{-13} and thus well below the numerical accuracy of the algorithm.

The results summarized in Fig. 3 confirm our expectation that the long range interaction (6.7) is only consistent with unitarity at a particular value of the dimensionless parameter s. Our best numerical determination of this critical value for the parameter is $s = s_{crit.} \sim 1.195 \pm .01$. A full solution of the model would *predict* a value for the dimensionless parameter s in the asymptotic interaction and furthermore relate the scale K to Λ_{ASP}. With the assumption that the approximations made in the evaluation of the expectation value of large Wilson loops are reasonable in CCG, the numerical evidence suggests that the string tension does not vanish and gives an estimate for $s_{crit.}$ if we require unitarity. Note that $s_{crit.}$ is a dimensionless parameter which would be a prediction of the gauge theory, whereas K is a scale, which essentially has to be determined experimentally. If the residue Z_L in (5.14) and the expectation value $\bar{\gamma}$ can be related to Λ_{ASP} in a two-loop calculation, a quantitative estimate of Λ_{ASP}/K eventually could be possible within the framework of perturbation theory.

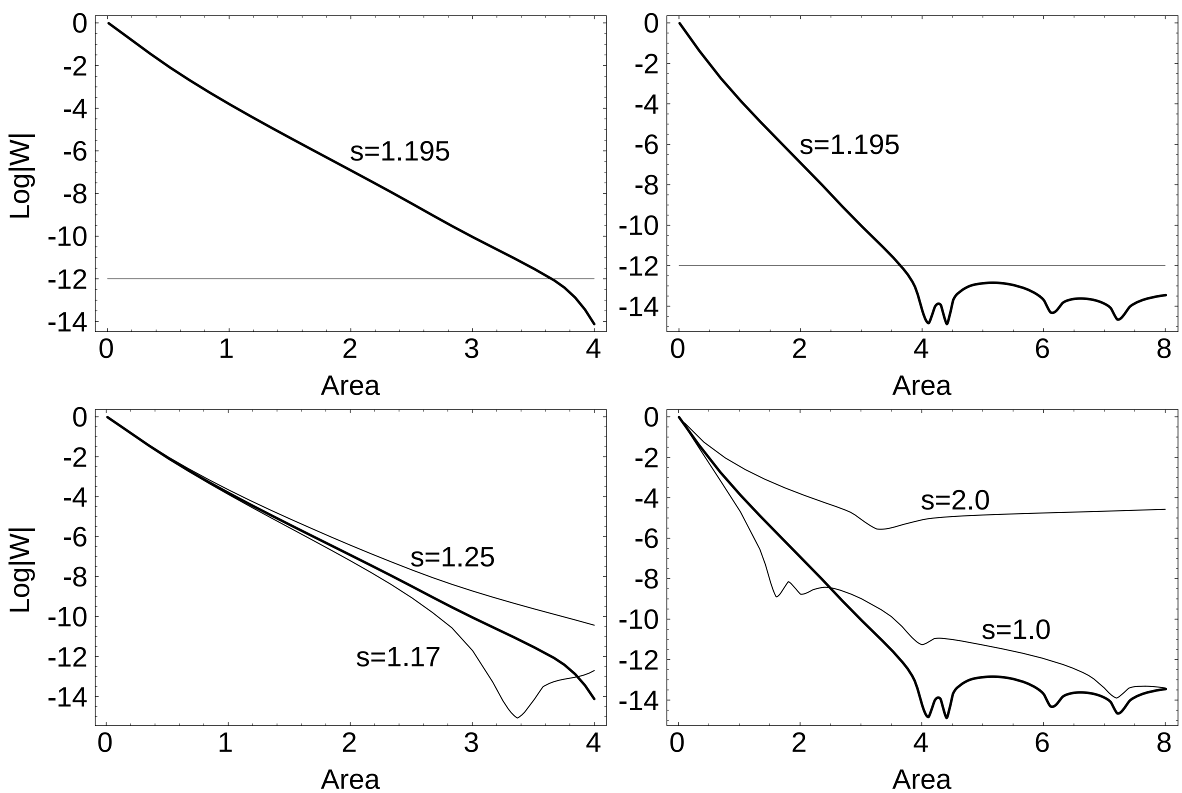

Fig. 3: The expectation value of large circular Wilson loops as a function of the enclosed dimensionless area $(KR)^2$ in the long range approximation of CCG.
Top-left: The logarithm of the expectation value for $s = 1.195$ in the range $(KR)^2 < 4$; the expectation value is positive in this range. The line at $\log(\langle W \rangle) = -12$ is an estimate of the

accuracy of the numerical solution.
Top-right: same as *top-left* but for the greater range $(KR)^2 < 8$; the plot shows the logarithm of the absolute value of the expectation value, which changes sign at the sharp dentures for $(KR)^2 > 4$. We attribute these oscillations of order 10^{-13} in the expectation value to numerical noise of the algorithm.
Bottom-left: The logarithm of the absolute value of the expectation value of the Wilson loop in the range $(KR)^2 < 4$ for parameter $s = 1.195$ (bold curve) as well as $s = 1.17$ and $s = 1.25$; for $s = 1.17$ the logarithm of the expectation value decreases faster than linearly with the area and violates unitarity.
Bottom-right: Same as *bottom-left* for a wider range of the parameter s and for $(KR)^2 < 8$. The result for $s = 2.0$ shows a violation of unitarity at the level of 10^{-4} which could be interpreted as due to a low-mass state of negative norm.

Our numerical studies also indicate that the Goldstone pole in (5.14) is essential for an exponential decrease of the Wilson loop in this approximation. The logarithmic dependence on the distance of the effective interaction (6.7) and (6.10) is a direct consequence of, and implies, a Goldstone pole in (5.14). We numerically investigated the effect of replacing the logarithm in (6.10) by a constant and found that this immediately leads to a non-unitary dependence of the expectation value of the Wilson loop. Our numerical results therefore depend sensitively on the asymptotic effective interaction (6.10) and are not a generic consequence of the non-linear integral equation (6.8). Partly due to this sensitivity however, we presently do not analytically understand the solution to this nonlinear integro-differential equation. An analytic, or approximate, solution to (6.11) for large values of $(KR)^2$ would provide a better understanding of this confinement mechanism in CCG.

7 Discussion

Since quarks have not been observed as asymptotic states, confinement is one of the central issues in $SU(n)$ gauge theories without fundamental scalars. On the other hand, these models are asymptotically free and perturbation theory in conjunction with the RG-equation is a powerful tool in their investigation. It was only recently observed that a translationally invariant BRST-quantization of these theories on compact space-time manifolds introduces a non-trivial moduli space[2, 9] because the BRST-symmetry has to be realized equivariantly with respect to global gauge transformations. In this work we found non-trivial fixed points on this moduli space in CCG, which respect the equivariant BRST-symmetry but break the global $SU(n)$-symmetry spontaneously to $U(1)^{n-1}$. The associated Goldstone bosons are $c\bar{c}$-bound states in the adjoint representation, which we argued lead to an IR-singular long distance behavior of the *longitudinal* gluon correlation of the form (5.14). Our numerical results for the Wilson loop indicate that this long range behavior of the longitudinal gluon propagator confines.

This scenario is however only realized in particular covariant gauges and only for $n_f \leq$

n quark flavors. In gauges other than CCG, the behavior of the Wilson loop is presumably similar, but the dynamics leading to confinement cannot be described in this way. More specifically, the relevant unphysical degrees of freedom that give a confining behavior of large Wilson loops in other covariant gauges are field configurations which cannot be described by the limited moduli-space at our disposal. The situation is somewhat analogous to atoms in QED: the spectrum of an atom is gauge invariant, but its qualitative description in gauges other than Coulomb gauge is a non-trivial matter. In Coulomb gauge, the unphysical degrees of freedom conspire to give a potential which describes the spectrum qualitatively and radiative corrections are computable and small. We believe that gauge freedom can also be exploited advantageously in $SU(n)$, but with the important difference that these non-abelian models are asymptotically free. Perturbation theory is valid only at high energies and we rely on spontaneous symmetry breaking to relate perturbative information to low energy dynamics. Within our limited moduli space, we can verify such a symmetry breakdown only in CCG and only for $n_f \leq n$. As far as we know, the expectation value of some composite scalar not described by our moduli space, would show that $SU(n)$ is broken to $U(1)^{n-1}$ also in covariant gauges other than CCG, and the corresponding Goldstone excitations could similarly lead to confinement. The limitation of our perturbative approach is that we cannot access the relevant composite scalars in gauges other than CCG. In non-covariant gauges the confinement mechanism is probably quite different and we have no suggestion how to perturbatively gain information about the low energy dynamics of the theory in this case.

The outstanding (and new) feature of the proposed confinement mechanism in CCG is the Goldstone pole in $\langle \partial\cdot A(x)\ \partial\cdot A(0)\rangle$, which implies that the *longitudinal* gluon interaction is proportional to $\sim 1/k^4$ for small momentum transfers. In conventional covariant gauges, the BRST-symmetry constrains the longitudinal propagator to be proportional to $1/k^2$; a $1/k^4$ behavior of the longitudinal propagator however does not violate Ward-identities in the extended covariant gauges with quartic ghost interaction ($\delta \neq 0$). We argued that perturbation theory is consistent with a dynamical spontaneous breaking of the global $SU(n)$-symmetry for certain critical gauges due to $ghost-antighost$-condensation in the adjoint. Remarkably, this spontaneous symmetry breaking of the gauge fixed theory does not give rise to a mass for the longitudinal degrees of freedom of the gauge field via the Higgs mechanism and the vertex (5.18) we obtain for the coupling between these modes does not vanish for $p^2 \to 0$. This is probably because the coupling of the longitudinal gauge field to the massless (composite) $c\bar{c}$-Goldstone modes is not constrained by gauge symmetry.

We presented numerical evidence that the observed behavior of the *longitudinal* propagator in non-abelian theories would confine in the sense of Wilson[4]. An appealing feature of this scenario is that the IR-singular behavior of the *longitudinal* exchange does not lead to a strong "Van der Waals" force between two distinct Wilson loops at large distances, because the interaction is a total derivative in both loop parameters. For sufficiently small loops, for which all exchanges within the loops can be neglected, the force between two Wilson loops due to longitudinal exchanges is therefore proportional to the physical size[17]

[17] Since the contribution from the exchange of one and two longitudinal gluons vanishes, the force is at

of either loop. The Van der Waals force from longitudinal exchange between *local* physical observables, for which this approximate evaluation of the long range interaction is valid in CCG, is thus seen to vanish.

Let us finally point out that the symmetry breaking we have discussed gives *quantitative* corrections to conventional perturbative results. These can in principle be calculated and experimentally verified. Of particular interest are perhaps power corrections to physical correlation functions at large Euclidean momentum transfers. These power corrections are described by vacuum expectation values of composite operators in the operator product expansion of Wilson[25]. At present these expectation values are determined experimentally or are obtained from lattice simulations. In CCG such power corrections arise naturally in the loop expansion due to the expectation values of the moduli.

Acknowledgments

We have greatly profited from extensive discussions with D. Zwanziger and L. Baulieu. This work could not have been finished without their support. We would like to thank D.Kabat and M.Porrati for explaining some aspects of confinement in supersymmetric gauge theories. A.R. enjoyed the hospitality of the LPTHE where this work was completed and was partially supported by a Margaret and Herman Sokol Research Fellowship.

A The partition function $\langle 1 \rangle$

We here adapt the proof[2] that the partition function of the TQFT of the gauge group does not vanish to our case. Consider the partition function

$$\mathcal{Z}[A] = \langle \mathbf{1} \rangle_A = \int [dU][dc][d\bar{c}][db] d\phi d\sigma d\bar{\sigma} d\bar{\gamma} d\gamma \; e^{S_A} \tag{A.1}$$

of the equivariant TQFT on the gauge group with action

$$S_A = sW_{GF}[A^U] \tag{A.2}$$

where $W_{GF}[A^U]$ is the gauge fixing functional (2.9) with A replaced by

$$A_\mu \to A_\mu^U = U^\dagger A_\mu U + \frac{1}{g} U^\dagger \partial_\mu U \tag{A.3}$$

A_μ in (A.1) is a background connection that identifies a particular orbit. $U(x) \in SU(n)$ is a local gauge transformation and the action of the BRST-operator s on the fields is defined as in (2.1) with the replacement (A.3). The action of s on the background connection A and the gauge transformation U individually is given by

$$sA_\mu(x) \;=\; 0$$

least proportional to the sixth power of the physical size of either Wilson loop

$$sU(x) \;=\; U(x)(gc(x)+\omega) \tag{A.4}$$

s thus effects an infinitesimal variation of the gauge group element U. It is straightforward to show that s is nilpotent ($s^2 = 0$). Because the action (A.2) is BRST-exact, (A.1) is the partition function of an equivariant TQFT of Witten type[26] on the gauge group. Standard arguments of TQFT show that (A.1) does not depend on the parameters α and δ in (2.9) and that (A.1) is proportional to the generalized Euler character of the space

$$\mathcal{E}[A] := \{U : U(x) \in SU(n), \partial\cdot A^U = 0\}/SU(n) \tag{A.5}$$

of solutions to the Landau gauge condition $\partial\cdot A = 0$ modulo global (right-hand) gauge transformations.

On the other hand the partition function $\langle \mathbf{1}\rangle$ of the equivariantly gauge fixed $SU(n)$ theory is proportional to

$$\langle \mathbf{1}\rangle = \int [\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A]\mathcal{Z}[A]e^{S_C} \tag{A.6}$$

where S_C is the classical gauge invariant action (2.6). Since the gauge symmetry is not anomalous, the change of variables

$$A^U \to A\ , \qquad U\psi \to \psi\ , \qquad \bar{\psi}U^\dagger \to \bar{\psi} \tag{A.7}$$

decouples the U-integration which is assumed to be normalized. The remaining path integral is just the one we are considering and the BRST-symmetry of the TQFT becomes the BRST-symmetry (2.1) of the model.

We show that $\mathcal{Z}[A]$ is a non-vanishing constant that does not depend on the background connection A. Since we are defining the model on a finite torus, an explicit mode expansion is possible. The Landau gauge condition is associated with fixed points of the Morse potential[6] $\mathcal{V}_A[U] \in \mathcal{R}_+ < \infty$

$$\mathcal{V}_A[U] = \int_{\mathcal{M}} \mathrm{Tr}\, A^U\cdot A^U \tag{A.8}$$

which is invariant with respect to global right-hand $SU(n)$ transformations. The generalized Euler characteristic is the (signed) sum of the Euler characters χ of the finite dimensional fixed point manifolds of (A.8) modulo $SU(n)$

$$\chi(\mathcal{E}[A]) = \sum_{\delta\mathcal{V}_A[U]|_{U\in\mathcal{M}_i}=0} \pm\chi(\mathcal{M}_i/SU(n)) \tag{A.9}$$

where the sign depends on the number of negative eigenvalues of the Hessian at a point of the fixed point manifold[26]. It is clear that this generalized Euler characteristic is constant under continuous deformations of the Morse potential $\mathcal{V}_A$. Since the space of $SU(n)$ connections on a (finite) torus is connected, $\mathcal{Z}[A]$ is thus a constant independent of A and it suffices to show that $\mathcal{Z}[A=0]$ does not vanish. To simplify matters, we only show that the generalized Euler characteristic of $\mathcal{E}[A=0]$ does not vanish for an $SU(2)$ gauge group on a torus.

The Morse potential $\mathcal{V}_{A=0}$ for $A=0$ is also symmetric with respect to *left* multiplication by a (constant) group element: $U(x) \to g_L U(x)$. Consequently, if $U(x)$ is a critical point of the potential (A.8) at $A=0$,

$$\partial \cdot (U^\dagger \partial U) = 0 \tag{A.10}$$

then so is

$$U(x, g_L, g_R) = g_L U(x) g_R, \qquad \forall g_L,\, g_R \in SU(n) \tag{A.11}$$

Some of these solutions however belong to the same equivalence class modulo right multiplication by global $SU_R(n)$ transformations. We obtain the moduli space of these (right equivalence classes of) solutions to (A.10) by noting that left multiplication of $U(x)$ by $g_L \in SU(n)$ is the same as right multiplication by

$$g_R(x) = U^\dagger(x) g_L U(x) \tag{A.12}$$

$g_L U(x)$ therefore belongs to the equivalence class modulo right multiplication of $U(x)$ by constant group elements only if $dg_R(x) = 0$, i.e.

$$[U(x) dU^\dagger(x),\, g_L] = 0 \tag{A.13}$$

Thus left multiplication of $U(x)$ by any $g_L \in SU(n)$ belonging to the subgroup which commutes with $U(x)dU^\dagger(x)$ is redundant. For an $SU(2)$ gauge group there are only three possible subgroups to consider:

$$\begin{aligned} &1)\ g_L \in SU_L(2) \text{ satisfy (A.13)} \Rightarrow \chi(SU_L(2)/SU_L(2)) = 1 \\ &2)\ g_L \in U(1) \subset SU_L(2) \text{ satisfy (A.13)} \Rightarrow \chi(SU_L(2)/U(1) \simeq S_2) = 2 \\ &3)\ g_L \in \{\mathbf{1}, -\mathbf{1}\} \subset SU_L(2) \text{ satisfy (A.13)} \Rightarrow \chi(SU_L(2)/\{\mathbf{1}, -\mathbf{1}\} \simeq SO(3)) = 0 \end{aligned} \tag{A.14}$$

Note that case 1) implies that $U(x)dU^\dagger(x) = 0$ and therefore corresponds to the equivalence class of the identity (modulo right multiplication). There is only *one* such class. For an $SU(2)$ gauge group, case 2) only occurs if $U(x)dU^\dagger(x) = d\theta(x)$ is an *abelian* connection. Since $U(x)$ should furthermore satisfy (A.10), we can conclude that this connection maps the 1-cycles of the torus to $U(1)$. A typical transformation of this kind on the torus has the form

$$U(x) = e^{a_\mu x_\mu}\ , \text{ with } [a_\mu,\, a_\nu] = 0 \tag{A.15}$$

with a constant abelian connection a_μ, whose components on a symmetric torus of extension L are

$$a_\mu = 2\pi i \hat{a} \cdot \vec{\sigma} k_\mu / L,\ k_\mu \in \mathbf{Z} \tag{A.16}$$

where the integers k_μ are the winding numbers of the mapping of the torus to the $U(1)$ subgroup and $\hat{a}$ is a unit vector which is the same for all components. Note that the corresponding constant abelian connections

$$A_\mu(x) = \frac{1}{g} U^\dagger \partial_\mu U = \frac{1}{g} a_\mu \tag{A.17}$$

are the Gribov copies of $A = 0$ on a torus we discussed in section 2.1. They come in pairs $(a, -a)$ that are related by a discrete $\mathbf{Z}_2$ subgroup of the $SU_R(2)$-transformations. A direct consequence is that the expectation value $\langle a_\mu \rangle$ vanishes for the equivariantly gauge fixed $SU(2)$ theory on a torus.

From (A.14) we obtain that the moduli space of solutions to (A.10) of an $SU(2)$ theory has the topological structure

$$\mathcal{E}_{A=0} = SU_R(2) \times \left[\mathbf{1} + S_2 \times \mathcal{F} + SO(3) \times \widetilde{\mathcal{F}}\right] \tag{A.18}$$

In other words $\mathcal{E}_{A=0}/SU(2)$ can essentially be described as a single point and a collection of two- and three-dimensional spheres[7, 27]. Although we do not know the Euler numbers associated with the topological spaces $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, the structure (A.18), together with $\chi(S_2) = 2$ and $\chi(SO(3)) = 0$ suffices to obtain that on a torus

$$\chi(\mathcal{E}[A = 0]/SU(2)) = odd \neq 0 \tag{A.19}$$

In the vicinity of $A = 0$ the degeneracy with respect to left group multiplication is in general lifted, but the signed sum of Morse indices over the (then isolated) fixed points still gives the Euler number (A.19). Since the space of $SU(2)$ connections on a torus is connected, we see that the partition function (A.1) of the TQFT is a constant independent of A which does not vanish. To see that the partition function $\langle \mathbf{1} \rangle$ of the gauge fixed theory (A.6) can then also be normalized, we observe that the determinant of the Euclidean Dirac-operator is positive for non-vanishing quark masses in this vector-like theory.

B The one loop effective potential of $SU(n)$

We here find the dependence of the 1-loop effective potential on the moduli $\bar{\gamma}$ by extending the derivation of the effective potential[2] for an $SU(2)$ group to $SU(n)$. For reasons given in section 4, all moduli except $\bar{\gamma}$ have vanishing expectation values. We take space-time to be a symmetrical torus $L \times L \times \ldots \times L$ of D dimensions[18]. The classical contribution to $V_{eff}(\bar{\gamma})$ is the quadratic term $-Tr\bar{\gamma}^2/\alpha g^2$ in the action (2.11). The g-independent term of next order in the loop expansion is obtained by evaluating the infinite sum of one ghost loop diagrams shown in Fig. 4. Crucial for the following is the absence of a compensating gluonic one-loop contribution to the effective potential, because the direct coupling $\int_{\mathcal{T}} \frac{1}{\alpha g}\bar{\gamma}\partial\cdot A$ of $\bar{\gamma}$ to the longitudinal gluon field in (2.11) vanishes on a torus with periodic boundary conditions for the gauge field.

[18]Apart from the finite volume we essentially follow the procedure of ref.[28] using dimensional regularization and the MS-scheme

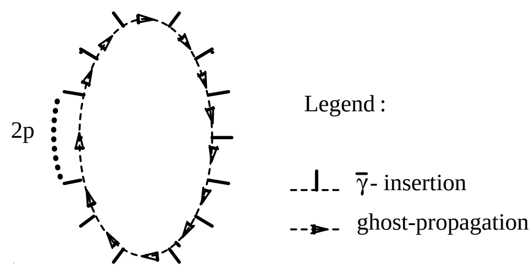

Fig. 4: Nonvanishing one-loop contribution to the effective moduli action: a ghost loop with $2p$ insertions of $\bar{\gamma}$-moduli.

Note that the antisymmetry of the structure constants f^{abc} implies that only loops with an even number of $\bar{\gamma}$-insertions give a non-vanishing contribution to the effective potential. The loop with $2p$ insertions of the ghost $\bar{\gamma}$ is proportional to

$$C_p = \mathrm{Tr}_{\,adj.}\hat{\bar{\gamma}}^{2p} \tag{B.1}$$

where we have used the notation $\hat{X}$ to denote the (anti-hermitian) matrix of the adjoint representation of X, $\hat{X}_{ab} = f^{abc}X^c$.

We want to express the trace (B.1) in terms of the eigenvalues of $\bar{\gamma}$ in the fundamental representation. The trace of interest is given by

$$\mathrm{Tr}_{\,adj.}\hat{\bar{\gamma}}^k = -2\mathrm{Tr}\ [[[\ldots[t^a, \underbrace{\bar{\gamma}]\ldots, \bar{\gamma}], \bar{\gamma}], \bar{\gamma}]}_{k \text{ times}}\, t^a \tag{B.2}$$

where t^a are the anti-hermitian generators of the fundamental representation of $SU(n)$ normalized to $\mathrm{Tr}\, t^a t^b = -\frac{1}{2}\delta^{ab}$. Using the Baker-Hausdorff formula the right hand side of (B.2) is written

$$\frac{d^k}{d\alpha^k}\mathrm{Tr}\ e^{-\bar{\gamma}\alpha}t^a e^{\bar{\gamma}\alpha}t^a\Big|_{\alpha=0} \tag{B.3}$$

We choose the basis in which $\bar{\gamma}$ is diagonal: $\bar{\gamma}_{kj} = i\lambda_k\delta_{kj}$ (no summation over k). The completeness relation for the t^a's, $t^a_{ij}t^a_{lm} = -\frac{1}{2}\left(\delta_{im}\delta_{lj} - \frac{1}{n}\delta_{ij}\delta_{lm}\right)$, gives:

$$\mathrm{Tr}\ e^{-\bar{\gamma}\alpha}t^a e^{\bar{\gamma}\alpha}t^a = \frac{1}{2} - \frac{1}{2}\mathrm{Tr}\ e^{-\bar{\gamma}\alpha}\mathrm{Tr}\ e^{\bar{\gamma}\alpha}$$

The evaluation of the derivatives of this expression for $p > 0$ leads to the desired expression of C_p in terms of the eigenvalues λ_i

$$\begin{aligned} C_p &= -2\frac{d^{2p}}{d\alpha^{2p}}\mathrm{Tr}\ e^{-\bar{\gamma}\alpha}t^a e^{\bar{\gamma}\alpha}t^a|_{\alpha=0} = \sum_{k=0}^{2p} C^k_{2p}(-1)^k \mathrm{Tr}\,\bar{\gamma}^k \mathrm{Tr}\,\bar{\gamma}^{2p-k} \\ &= \sum_{k=0}^{2p} C^k_{2p}(-1)^{p+k}\sum_i^n \lambda_i^k \sum_j^n \lambda_j^{2p-k} = 2(-1)^p \sum_{1\le i<j\le n}(\lambda_i - \lambda_j)^{2p} \end{aligned} \tag{B.4}$$

The contribution to the effective potential from a single loop with $2p$ insertions of the moduli is

$$F_D(p) = \frac{1}{2pL^D}\mathrm{Tr}_{\,adj.}\left(\frac{L^2\delta\hat{\bar{\gamma}}}{(2\pi)^2}\right)^{2p}\sum_{\{n_1\ldots n_D\}\neq\{0\ldots0\}}\left(n_1^2 + \ldots + n_D^2\right)^{-2p} \tag{B.5}$$

The sum in (B.5) extends over all the sets of integers $\{n_1 \dots n_D\}$ describing the complete set of modes with momenta $k_\mu = 2\pi n_\mu/L$ of the dynamical ghosts. The contribution from the constant modes with $n_\mu = 0,\ \mu = 1, \dots, D$, to (B.5) is however eliminated by the integration over the ghosts $\bar\sigma$ and γ in (2.11). $F_D(p)$ is therefore finite for $p > D/4$. The overall sign of (B.5) is due to ghost statistics.

One can analytically continue to non-integer dimensions by casting $F_D(p)$ in integral form,

$$L^D F_D(p) = \mathrm{Tr}_{\,adj.} \left(\frac{L^2\delta\hat{\bar\gamma}}{(2\pi)^2}\right)^{2p} \frac{1}{\Gamma(2p+1)} \int_0^\infty dx\, x^{2p-1} \left[(f(x))^D - 1\right] \tag{B.6}$$

where the function $f(x)$ for $x > 0$ is the convergent sum,

$$f(x) = \sum_{n=-\infty}^{\infty} e^{-xn^2} \tag{B.7}$$

Although there is no analytic expression for $f(x)$ at arbitrary values of its argument, the reflection formula[29]

$$f(x) = \sqrt{\frac{\pi}{x}}\ f(\frac{\pi^2}{x}) \tag{B.8}$$

implies the asymptotic behavior,

$$\begin{aligned} f(x \to \infty) &= 1 + 2e^{-x} + O(e^{-4x}) \\ f(x \to 0) &= \sqrt{\frac{\pi}{x}}(1 + O(e^{-\frac{\pi^2}{x}})) \end{aligned} \tag{B.9}$$

Since the eigenvalues of the commuting ghost $\bar\gamma$ can be treated as real numbers

$$v_{ij}^2 = \frac{(\lambda_i - \lambda_j)^2\delta^2}{(4\pi)^2\mu^4} \tag{B.10}$$

is positive and the 1-loop contributions to the effective potential can be summed over all values of $p \geq 1$. In (B.10) we have introduced the *finite* scale μ because we are interested in the $\mu L \to \infty$ limit and the eigenvalues λ_i in this limit are measured in terms of the finite renormalization scale μ. Summing over all one loop contributions and adding the tree-level potential $-\mathrm{Tr}\,\bar\gamma^2/\alpha g^2 = \sum_{i<j}(4\pi)^2\mu^4 v_{ij}^2/n\alpha\delta^2 g^2$ gives

$$V_{1-loop} = \sum_{1\leq i<j\leq n} \mu^4 \frac{(4\pi)^2 v_{ij}^2}{n\alpha\delta^2 g^2} - 4L^{-D} \int_0^\infty \frac{dx}{x} \sin^2(v_{ij}x/2)) \left[\left(f(\pi x/(L\mu)^2)\right)^D - 1\right] \tag{B.11}$$

The asymptotic behavior of $f(x)$ for $x \to 0$ determines the leading behavior of the integral in (B.11) when $L\mu \to \infty$. This is the large-volume limit we are interested in. For $L\mu \to \infty$ one thus obtains in $D = 4 - 2\varepsilon$ dimensions,

$$V_{1-loop} = \sum_{1\leq i<j\leq n} \mu^D v_{ij}^2 \left[\frac{(4\pi)^2\mu^{2\varepsilon}}{n\alpha\delta^2 g^2} - 2v_{ij}^{-\varepsilon}\cos(\frac{1}{2}\pi\varepsilon)\Gamma(\varepsilon - 2)\right] \tag{B.12}$$

The $1/\varepsilon$-term in the expansion of (B.12) is canceled by a counter term $(1-Z)L^D \mathrm{Tr}\, \frac{\bar{\gamma}^2}{\alpha g^2}$ in the classical action. To order $\hbar$, we require in the MS-scheme,

$$Z = 1 + \frac{n\delta^2 \alpha g^2 \mu^{-2\varepsilon}}{16\pi^2 \varepsilon} \tag{B.13}$$

Introducing the scale κ

$$\ln \frac{\kappa^2}{4\pi\mu^2} = -\frac{(4\pi)^2}{n\alpha\delta^2 g^2} - \gamma_E + 1 + O(\ln g^2, g^2), \tag{B.14}$$

the expression (B.12) for the 1-loop effective potential in $D = 4$ dimensions can be written

$$V_{1-loop} = \sum_{1\le i<j\le n} \frac{(\lambda_i - \lambda_j)^2 \delta^2}{32\pi^2} \ln \left[\frac{(\lambda_i - \lambda_j)^2 \delta^2}{e\kappa^4} \right] \tag{B.15}$$

which is (4.4) in section 4.

The counter term of the classical potential together with Z_α and Z_g given in (3.2), also determines the renormalization constant $Z_{\bar{\gamma}^2}$ of the global ghost $\bar{\gamma}$ to order g^2. Using (B.13) we obtain in the MS-scheme

$$Z_{\bar{\gamma}^2} = Z Z_\alpha Z_g^2 = 1 - \frac{n g^2 \mu^{-2\varepsilon}}{32\pi^2 \varepsilon}(3 + \alpha(1 - 2\delta)) + O(g^4) \tag{B.16}$$

Note that a derivation of the critical exponent of $\bar{\gamma}$ via the effective potential only involves UV-divergences and that IR-divergences are absent from this calculation.

C One-loop contribution to the vertex (5.18)

We are interested in the color antisymmetric part of the mixing between the composite field $\mathcal{O}^a$ given in (5.8) and $\partial{\cdot}A$ for vanishing momentum transfer. The one-loop contribution to the vertex (5.18) is obtained by evaluating the infinite sum of ghost loop diagrams of Fig. 5

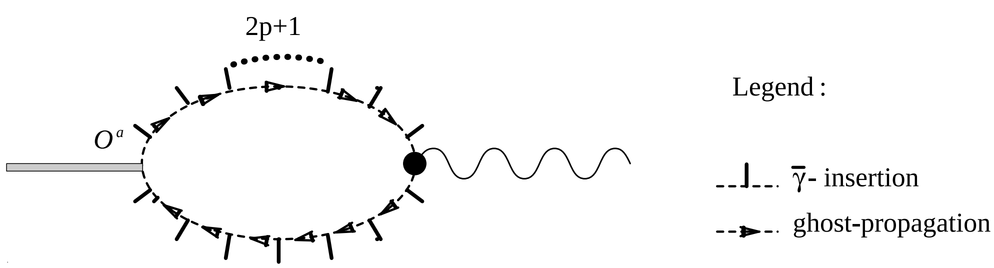

Fig. 5: Nonvanishing one-loop contribution to the vertex (5.18).

Contributions antisymmetric in the color indices arise only from diagrams with an odd number of insertions of the matrix $\hat{\bar{\gamma}}_{ab} = f^{abc}\bar{\gamma}^c$. For simplicity, we evaluate this

contribution to the vertex for an $SU(2)$ gauge group. In a parametric form suitable for performing the loop integral, the effective ghost propagator in momentum space is

$$\begin{aligned}
G_{ab}(p^2) &\equiv \frac{1}{p^2\mathbf{1}+\delta\hat{\bar{\gamma}}} = \int_0^\infty d\lambda e^{-\lambda(p^2\mathbf{1}+\delta\hat{\bar{\gamma}})} \\
&= \int_0^\infty d\lambda e^{-\lambda p^2}\sum_{j=0}^{\infty}\frac{(-\delta\lambda\hat{\bar{\gamma}})^j}{j!} \\
&= \int_0^\infty d\lambda e^{-\lambda p^2}\left[\frac{\delta^2\bar{\gamma}^a\bar{\gamma}^b}{\kappa^4}+\bar{\gamma}_\perp^{ab}\cos(\lambda\kappa^2)+\delta\frac{\varepsilon^{abc}\bar{\gamma}^c}{\kappa^2}\sin(\lambda\kappa^2)\right] \qquad \text{(C.1)}
\end{aligned}$$

where

$$\bar{\gamma}_\perp^{ab} = \delta^{ab}-\delta^2\frac{\bar{\gamma}^a\bar{\gamma}^b}{\kappa^4} = -\delta^2\frac{\hat{\bar{\gamma}}_{ac}\hat{\bar{\gamma}}_{cb}}{\kappa^4} \qquad \text{(C.2)}$$

projects on color degrees of freedom transverse to $\bar{\gamma}$ and $\kappa^4=\delta^2\bar{\gamma}^a\bar{\gamma}^a$ is the scale of SSB of the effective potential (4.4) for an $SU(2)$ theory. The fact that the summation in (C.1) gives a relatively simple expression for an $SU(2)$ group, greatly facilitates the evaluation of the one-loop contribution to the vertex. Note that the poles of the Euclidean ghost propagator (C.1) are all at purely imaginary p^2 and that the massless ghost is precisely that of the unbroken $U(1)$. The sum of Feynman diagrams with p insertions of $\hat{\bar{\gamma}}$ in D space-time dimensions corresponds to the loop-integral

$$\Gamma_\mu^{ab}(q) = iq_\mu\Gamma^{ab}_{\mathcal{O}\,\partial A}(q^2) = -ig^2\alpha\delta\int\frac{d^Dk}{(2\pi)^D}\,\varepsilon^{cad}G_{de}(k^2)\varepsilon^{ebg}G_{gc}((k-q)^2))(\delta k_\mu+(1-\delta)(k-q)_\mu) \qquad \text{(C.3)}$$

Using the parameterization (C.1) for the ghost propagator, performing the color summations and retaining only the part of $\Gamma^{ab}_{\mathcal{O}\,\partial A}$ that is antisymmetric in the color indices one obtains from (C.3)

$$\Gamma^{[ab]}_{\mathcal{O}\,\partial A}(q^2) = g^2\alpha(2\delta-1)\delta^2\frac{\varepsilon^{abc}\bar{\gamma}^c}{\kappa^2}\int\frac{d^4k}{(2\pi)^4}\int_0^\infty d\lambda\sin(\lambda\kappa^2)\int_0^\infty d\lambda' e^{-\lambda k^2-\lambda'(k-q)^2} \qquad \text{(C.4)}$$

where we have already taken the limit $D\to 4$ since the integral (C.4) is UV and IR finite. The vertex in the limit of vanishing momentum transfer $q^2\to 0$ is particularly simple to evaluate

$$\begin{aligned}
\Gamma^{[ab]}_{\mathcal{O}\,\partial A}(q^2\to 0) &= \varepsilon^{abc}\bar{\gamma}^c\frac{g^2\alpha(2\delta-1)\delta^2}{16\pi^2\kappa^2}\int_0^\infty d\lambda\frac{\sin(\kappa^2\lambda)}{\lambda} \\
&= \varepsilon^{abc}\bar{\gamma}^c\frac{g^2\alpha(2\delta-1)\delta^2}{32\pi\kappa^2} \qquad \text{(C.5)}
\end{aligned}$$

Using that $\alpha(2\delta-1)=3$ in CCG, the last expression in (C.5) for the antisymmetric vertex at vanishing momentum transfer becomes (5.18) of the main text.

D RG-invariance of the effective long-range interaction

We here determine the renormalization of the effective interaction (6.7) to order g^2 for planar contributions to large Wilson loops of the kind shown in Fig. 2. Power counting

suggests that UV-divergences do not arise from this effective interaction since the coupling K has positive mass dimension. To lowest order in $g(\mu)$ the interaction (6.7) is just the product of the gluon propagator and the coupling g^2. This interaction therefore depends on the renormalization point μ. We will show that this μ-dependence of the lowest order interaction is however canceled to order g^2 by vertex corrections in the Wilson loop.

The Wilson loop itself is a RG-invariant[30] observable, and can therefore be expressed in terms of the bare coupling g_B and the bare gluonic field A_B which do not depend on the renormalization scale. The bare field and coupling are related to the corresponding renormalized quantities by (3.1), with the renormalization constants of (3.2). The product of propagator and coupling constant g^2 renormalizes with $Z_g^2 Z_3$

$$g_B^2 \langle A_B(x)A_B(0)\rangle = Z_g^2 Z_3 g^2 \langle A(x)A(0)\rangle \tag{D.1}$$

(D.1) is proportional to the effective 2-point interaction (6.3) in the Wilson loop however only to *lowest* order in g^2. In order g^4 the divergent parts of the diagrams in Fig. 6 also contribute to the effective 2-point interaction (6.7). As we will see, these divergences compensate the factor $Z_g^2 Z_3$ in (D.1) to order g^2. This implies that the (finite) corrections from the contributions of Fig. 6 precisely cancel the μ dependence of the renormalized effective 2-point interaction to order g^2. The anomalous dimension of the effective 2-point interaction (6.7) is then at least of order g^4, which is sufficient to prove that it is renormalization group invariant in the limit $\mu \to \infty$.

We show that the divergent corrections from planar diagrams of order g^2 shown in Fig. 6 can be absorbed in the *longitudinal* interaction by a multiplicative factor

$$(Z_g^2 Z_3)^{-1} = 1 + \frac{g^2 n \mu^{-2\varepsilon}}{32\pi^2\varepsilon}(3+\alpha) + O(g^4) \tag{D.2}$$

if the lowest order interaction is longitudinal (i.e. we consider only $O(g^2)$ corrections to the long-range interaction).

Only the self-energy insertions (Fig. 6a) and vertex correction (Fig. 6b) are UV-divergent in dimensional regularization as $\varepsilon = (4-D)/2 \to 0$. To evaluate the divergences, only the behavior of propagators at high momenta are needed and perturbative propagators and vertices are therefore sufficient in an asymptotically free model.

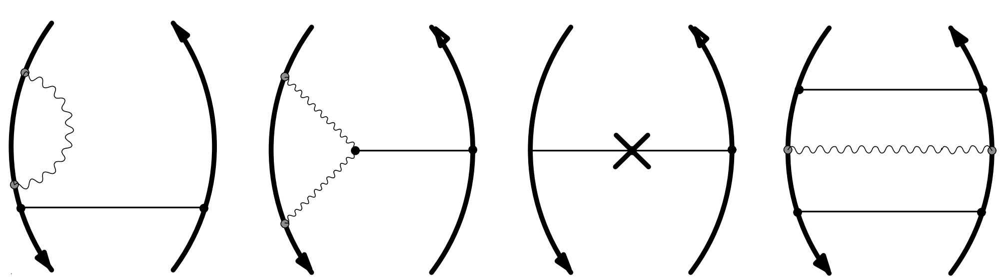

Fig. 6: Planar corrections of order g^2 to the effective longitudinal interaction. Sections of the Wilson loop are shown in bold and the path ordering is indicated by arrows; the effective longitudinal interaction is depicted as a thin line and perturbative gluon exchanges as wavy

lines. *a)* A self-energy insertion on the Wilson line. *b)* A perturbative vertex corrections to the long-range longitudinal gluon interaction. *c)* Counter-term $Z_g^2 Z_3 - 1$ due to renormalization of the effective longitudinal interaction. *d)* A finite perturbative correction of order g^2.

Let us first analyze the vertex correction of Fig. 6b). In the Wilson loop this correction gives rise to a loop integral of the form

$$\begin{aligned} p_\omega \Gamma^a_\omega(p)|_{Fig.6b} &= -ig^2 \frac{n}{2} t^a \int\int d\theta\, d\theta' \frac{dx_\mu}{d\theta} \frac{dy_\nu}{d\theta'} \Theta(\theta-\theta') \int \frac{d^D k}{(2\pi)^D} e^{i(k(x-y)-py)} \times \\ &\times D_{\mu\rho}(k) \left[\delta_{\rho\sigma}(k^2-(p+k)^2) - k_\rho k_\sigma + (p+k)_\rho (p+k)_\sigma\right] D_{\sigma\nu}(p+k) \end{aligned} \tag{D.3}$$

where we have used that the lowest order effective interaction is longitudinal. In (D.3) the color traces have been performed, $D_{\mu\nu}(p)$ denotes the perturbative gluon propagator and $x(\theta), y(\theta')$ are points on the Wilson loop. The Θ-function in (D.3) arises due to path ordering. Note that the term in square brackets of (D.3) is a linear superposition of transverse projectors. Inserting the explicit expression for the perturbative gluon propagator, the integrand of (D.3) can be rewritten as

$$\begin{aligned} &D_{\mu\rho}(k) \left[\delta_{\rho\sigma}(k^2-(p+k)^2) - k_\rho k_\sigma + (p+k)_\rho (p+k)_\sigma\right] D_{\sigma\nu}(p+k) = D_{\mu\nu}(p+k) - \\ &-D_{\mu\nu}(k) + \frac{k_\mu p_\nu + p_\mu (p+k)_\nu}{(p+k)^2 k^2} + (\alpha-1)(k_\mu (p+k)_\nu (k^2+pk) \frac{(k^2+pk)(p^2+2pk)}{k^4 (p+k)^4} \end{aligned} \tag{D.4}$$

The first two terms in the last expression for the integrand of (D.3) precisely compensate the divergent contributions from the self-energy insertions[19] of Fig. 6a). The other terms in (D.4) are longitudinal in one or the other (or both) gluon vertices. The μ-dependent divergent contributions of order g^2 from the vertex- *and* self-energy corrections to the effective interaction therefore arise from the divergent part of the integral

$$\begin{aligned} p_\omega \Gamma^a_\omega(p)|_{Fig.6a+b} &= \text{convergent} + \frac{g^2 n}{2} t^a \int\int d\theta\, d\theta'\, \Theta(\theta-\theta') \frac{dx_\mu}{d\theta} \frac{dy_\nu}{d\theta'} \times \\ &\times \int \frac{d^D k}{(2\pi)^D} \left[\frac{1}{(k-p/2)^2 (k+p/2)^2} \left(p_\mu \frac{\partial}{\partial y_\nu} - p_\nu \frac{\partial}{\partial x_\mu}\right) + \right. \\ &\left. +(\alpha-1) \frac{pk(k^2-p^2/4)}{(k+p/2)^4 (k-p/2)^4} \left((k-p/2)_\mu \frac{\partial}{\partial y_\nu} - (k+p/2)_\nu \frac{\partial}{\partial x_\mu}\right)\right] e^{i[(k-p/2)x-(k+p/2)y]} \end{aligned}$$
$$(D.5)$$

Observing that the remaining integral is divergent only for $x = y$, one can now perform one of the line integrals and obtains

$$p_\omega \Gamma^a_\omega(p)|_{Fig.6a+b} = \text{convergent} + g^2 n t^a \int d\theta \frac{dx_\mu}{d\theta} e^{-ipx} p_\mu \times$$

[19]Note that the long-range interaction in Fig. 6a) is longitudinal – μ-dependent divergences arise only from so-called "pinch" contributions[31] with at least two coinciding vertices. Apart from a difference in the overall sign, the integrands of these "pinched" self-energy contributions are just the first two terms in (D.4).

$$
\begin{aligned}
&\times \int \frac{d^D k}{(2\pi)^D} \left[\frac{1}{(k-p/2)^2(k+p/2)^2} + \frac{\alpha - 1}{D} \frac{k^2(k^2 - p^2/4)}{(k+p/2)^4(k-p/2)^4} \right] \\
= \quad & \text{convergent} + \frac{g^2 \mu^{-2\varepsilon} n}{64\pi^2 \varepsilon}(3+\alpha) t^a \int d\theta \, \frac{dx_\mu}{d\theta} e^{-ipx} p_\mu \qquad \text{(D.6)}
\end{aligned}
$$

as the divergent part of the vertex and self-energy corrections to the effective longitudinal interaction [20] in order g^2. The divergent contributions we calculated arise at both vertices of the effective interaction and thus precisely cancel the counter-term $Z_g^2 Z_3 - 1$ from the renormalization of the lowest order effective interaction to order g^2. In the Wilson loop, the μ-dependence of the effective longitudinal interaction is therefore compensated by the μ-dependence of vertex and self-energy corrections to order g^2.

Only the perturbative diagrams considered here are planar and UV-divergent to order g^2. It might seem that IR-divergent contributions arise from the exchange of longitudinal perturbative gluons shown in Fig. 6d). This is however not the case and the IR-divergences in fact cancel. To order g^2 the effective longitudinal 2-point interaction is independent of μ and its anomalous dimension is therefore of order g^4. Perturbative corrections to the effective long range interaction are therefore analytic in g and K and s in (6.7) approach RG-invariant constants as $g \to 0$.

E Integral equation for a planar Wilson loop with longitudinal interaction

In the following we derive the integral equation (6.8) satisfied by a Wilson loop with an effective longitudinal gluonic 2-point interaction (6.7). As has been argued in section 6, we need to consider only contributions which are of leading order in an $1/n$ expansion – for color-vector interactions, these are planar diagrams with all exchanges inside the loop.

Consider a typical contribution to the Wilson loop with only 2-point interactions of this type. We open the loop at some point A as indicated in Fig. 7. Since the vertices on the loop are *ordered* and the exchanges are planar in the above sense, there is a one-to-one correspondence between one-sided planar contractions of vertices on a "line" and corresponding contributions to the Wilson loop if each topologically different one-sided planar diagram of the "line" is counted exactly once. Although several topologically different diagrams on the "line" may correspond to the same diagram for the Wilson loop, this multiplicity precisely accounts for the statistical factor of that diagram in the loop. We are thus left with the combinatorial problem of generating all topologically different one-sided planar contributions to the "line" exactly once.

[20] Note that for a *transverse* gluonic interaction the divergent perturbative corrections could not be absorbed multiplicatively, since the operators $A_{[\mu,\nu]}$ and $[A_\mu, A_\nu]$ mix in order g^2.

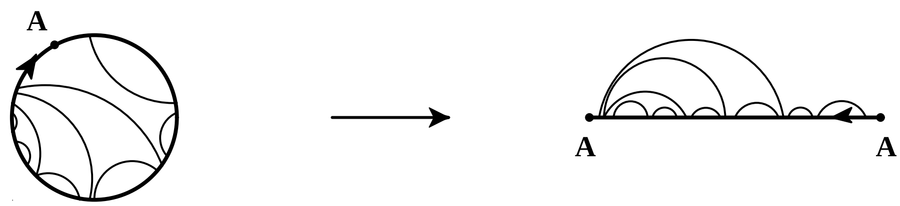

Fig. 7: A planar contribution to the Wilson loop from a 2-point interaction and the corresponding one-sided planar "line"-diagram when the loop is opened at A

The closed curve of the Wilson loop $\mathcal{C} := \{x_\mu(\theta) : \theta \in [0, 2\pi] \text{ with } x_\mu(0) = x_\mu(2\pi)\}$, is parameterized by $\theta \in [0, 2\pi]$. In the following we reserve Greek letters for the one-dimensional coordinates on the loop respectively "line". An effective longitudinal interaction of the form $v^{ab}_{\mu\nu}(x-y) = \delta^{ab}\partial_\mu\partial_\nu\sigma((x-y)^2)/n$ in leading order of $1/n$ results in a scalar factor

$$v(\alpha, \beta) = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial\beta}\sigma((x(\alpha) - y(\beta))^2) \tag{E.1}$$

for each interaction on the "line" between points α and β of such a one-sided planar diagram, since the color matrices at each vertex always contract to give the quadratic Casimir of the fundamental representation. In leading order of $1/n$, the "line" is therefore proportional to the unit matrix. Note that for a *transverse* exchange the interaction depends on the *direction* relative to the line element and is not of the form (E.1). For an abelian group there is no distinction between the ordering of the vertices and *any* longitudinal interaction gives a vanishing contribution to the Wilson loop, since (E.1) is a total derivative in the loop parameter. In non-abelian gauge theories, the ordering of the vertices does however matter and longitudinal interactions do contribute to the Wilson loop. Without the spontaneous symmetry breaking, the contribution from longitudinal 2-point interactions would be compensated by contributions from Green's functions of higher order. As explained in the main text, we however do not expect such a compensation of the long range longitudinal interaction when the global $SU(n)$-symmetry is spontaneously broken.

Let us denote by $F_k(\alpha, \beta)$ the sum of all one-sided planar diagrams containing k longitudinal interactions between the points α and β on the "line", with each topologically distinct diagram counted exactly once. Diagrammatically, $F_{k+1}(\alpha, \beta)$ is related to $F_l(\alpha, \beta)$ with $l \leq k$ by:

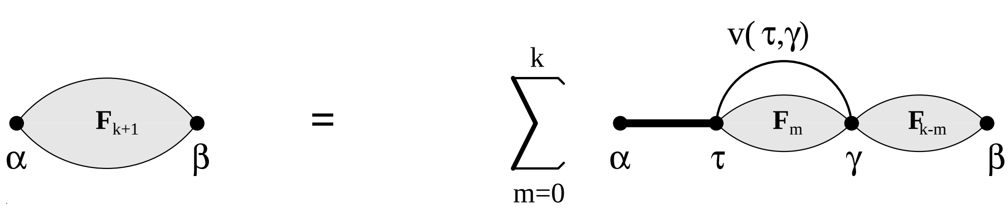

Fig. 8: Iterative equation for $F_{k+1}(\alpha, \beta)$.

Fig. 9 expresses the fact that the first interaction in a one-sided planar diagram of order $k+1$ "encloses" a one-sided planar diagram of order less than k and that the remaining

diagram after this first interaction is one-sided planar and of the complimentary order. Naturally, one has to sum over all intermediate orders and integrate over all possible (ordered) positions of the vertices for the first interaction between the points α and β. It is not hard to show inductively that each topologically distinct one-sided planar diagram of order $k+1$ appears exactly once in F_{k+1}. Written explicitly, the one-sided planar contribution of order $k+1$ therefore is,

$$F_{k+1}(\alpha,\beta) = \sum_{m=0}^{k} \int_{\alpha}^{\beta} d\gamma \int_{\alpha}^{\gamma} d\tau\, v(\tau,\gamma) F_m(\tau,\gamma) F_{k-m}(\gamma,\beta) \tag{E.2}$$

where $F_0(\alpha,\beta) = 1$ is the contribution with no interactions. The total one-sided planar contributions $W(\alpha,\beta)$ to the "line" between α and $\beta \geq \alpha$ is just the sum of all F_k, since each has the statistical factor 1:

$$W(\alpha,\beta) = \sum_{k=0}^{\infty} F_k(\alpha,\beta) = F_0(\alpha,\beta) + \sum_{k=0}^{\infty} F_{k+1}(\alpha,\beta) \tag{E.3}$$

Using (E.2) and rearranging the double sum, one obtains

$$\begin{aligned} W(\alpha,\beta) &= F_0(\alpha,\beta) + \sum_{k=0}^{\infty}\sum_{m=0}^{k} \int_{\alpha}^{\beta} d\gamma \int_{\gamma}^{\beta} d\tau\, v(\tau,\gamma) F_m(\tau,\gamma) F_{k-m}(\gamma,\beta) \\ &= 1 + \int_{\alpha}^{\beta} d\gamma \int_{\alpha}^{\gamma} d\tau\, v(\tau,\gamma) W(\tau,\gamma) W(\gamma,\beta) \\ &= 1 + \int_{\alpha}^{\beta} d\tau \int_{\tau}^{\beta} d\gamma\, v(\tau,\gamma) W(\tau,\gamma) W(\gamma,\beta). \end{aligned} \tag{E.4}$$

Relatively simple is the case of a circular Wilson loop parameterized by an angle with a longitudinal interaction of the form (E.1). The distance between points on the circular loop in this case depends only on the difference in their angular positions and the radius R: $(x-y)^2 = 4R^2 \sin^2\left(\frac{\theta_x - \theta_y}{2}\right)$. For a longitudinal and translationally and rotationally invariant interaction, $v(\alpha,\beta) = v_R(\alpha-\beta)$ of (E.1) in this case is a function of the angular difference $\alpha-\beta$ and the radius only. Thus, for a circle of radius R, $W(\alpha,\beta) = W_R(\alpha-\beta)$ also depends only on the angular difference and one can rewrite (E.4) as:

$$\begin{aligned} W_R(\theta) &= 1 + \int_0^{\theta} d\theta' \int_{\theta'}^{\theta} d\theta''\, v_R(\theta''-\theta') W_R(\theta''-\theta') W_R(\theta-\theta'') \\ &= 1 + \int_0^{\theta} d\theta' \int_0^{\theta'} d\theta''\, v_R(\theta'') W_R(\theta'') W_R(\theta'-\theta''). \end{aligned} \tag{E.5}$$

To solve (E.5) numerically, it is advantageous to consider instead the equivalent non-linear integro-differential equation:

$$\frac{d}{d\theta} W_R(\theta) = \int_0^{\theta} d\theta'\, v_R(\theta') W_R(\theta') W_R(\theta-\theta'), \tag{E.6}$$

with the initial value $W_R(0) = 1$. The expectation value of the Wilson loop in this approximation is then $nW_R(2\pi)$.

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